
Sir Timothy Waldo

Keill, John

Euclid's ELEMENTS
OF
GEOMETRY,

From the *Latin*
Translation of COMMANDINE.

To which is added,

A TREATISE of the *Nature* and *Arithmetick* of LOGARITHMS; Likewise
Another of the ELEMENTS of *Plain* and *Spherical* TRIGONOMETRY; With
A PREFACE, shewing the *Usefulness* and *Excellency* of this WORK:

By Doctor JOHN KEIL, *F. R. S.*
and late Professor of ASTRONOMY *in* Oxford.

Now done into ENGLISH.

The *Whole* revis'd; where deficient, supplied; where lost or corrupted, restor'd. Also
Many Faults committed by Dr. *Harris*, Mr. *Caswel*, Mr. *Heynes*, and other *Trigonometrical Writers*, are shewn; and in those Cases where *They* are mistaken, here are given Solutions *Geometrically* true.
A more Ample Account of which may be seen in Mr. *Cunn*'s PREFACE.

By Mr. SAMUEL CUNN.

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Dr. *KEIL*'s

PREFACE.

A *YOUNG Mathematician may be ſurprized, to ſee the old obſolete* Elements *of* Euclid *appear afreſh in Print; and that too after ſo many new Elements of Geometry, as have been lately publiſh'd; eſpecially ſince thoſe who gave us the Elements of Geometry, in a new Manner, would have us believe they have detected a great many Faults in* Euclid. *Theſe acute* Philoſophers, *pretend to have diſcover'd that* Euclid's *Definitions are not perſpicuous enough; that his Demonſtrations are ſcarcely evident; that his whole*

Elements are ill dispos'd; and that they have found out innumerable Falsities in them, which had lain hid to their Times.

But by their Leave, I make bold to affirm, that they Carp at Euclid *undeservedly: For his Definitions are destinct and clear, as being taken from first Principles, and our most easy and simple Conceptions; and his Demonstrations elegant, perspicuous and concise, carrying with them such Evidence, and so much Strength of Reason, that I am easily induced to believe the Obscurity,* Sciolists *so often accused* Euclid *with, is rather to be attributed to their own perplex'd Ideas, than to the Demonstrations themselves. And however some may find Fault with the Disposition and Order of his Elements, yet notwithstanding I do not find any Method, in all the Writings of this kind, more proper and easy for Learners than that of* Euclid.

It is not my Business here to Answer separately every one of these Cavellers; *but it will easily appear to any one, moderately versed in these Elements, that they*

they rather shew their own Idleness, than any real Faults in Euclid. *Nay, I dare venture to say, there is not one of these New Systems, wherein there are not more Faults, nay, grosser Paralogisms, than they have been able even to imagine in* Euclid.

After so many unsuccessful Endeavours, in the Reformations of Geometry, some very good Geometricians, not daring to make new Elements, have deservedly preferr'd Euclid *to all others; and have accordingly made it their Business to publish those of* Euclid. *But they, for what Reason I know not, have entirely omitted some Propositions, and have altered the Demonstrations of others for worse. Among whom are chiefly* Tacquet *and* Dechalles, *both of which have unhappily rejected some elegant Propositions in the Elements, (which ought to have been retain'd,) as imagining them trifling and useless; such, for Example, as* Prop. 27, 28, *and* 29. *of the Sixth Book, and some others, whose Uses they might not know. Farther, wherever they use Demonstrations of their own, in-*

stead of Euclid's, *in those Demonstrations they are faulty in their Reasoning, and deviate very much from the Conciseness of the Antients.*

In the fifth Book, they have wholly rejected Euclid's *Demonstrations, and have given a Definition of Proportion different from* Euclid's; *and which comprehends but one of the two Species of Proportion, taking in only commensurable Quantities. Which great Fault no Logician or Geometrician would have ever pardoned, had not those Authors done laudable Things in their other Mathematical Writings. Indeed, this Fault of theirs is common to all Modern Writers of Elements, who all split on the same Rock; and to shew their Skill, blame* Euclid, *for what, on the contrary, he ought to be commended; I mean, the Definition of Proportional Quantities, wherein he shews an easy Property of those Quantities, taking in both Commensurable and Incommensurable ones, and from which, all the other Properties of Proportionals do easily follow.*

Some Geometricians, forsooth, want a Demonstration of this Property in Euclid; *and*

and undertake to supply the Deficiency by one of their own. Here, again, they shew their Skill in Logick, in requiring a Demonstration for the Definition of a Term; that Definition of Euclid *being such as determines those Quantities Proportionals which have the Conditions specified in the said Definition. And why might not the Author of the Elements give what Names he thought fit to Quantities having such Requisites; surely he might use his own Liberty, and accordingly has called them Proportionals.*

But it may be proper here to examine the Method whereby they endeavour to Demonstrate that Property: Which is by first assuming a certain Affection, agreeing only to one kind of Proportionals, viz. *Commensurables; and thence, by a long Circuit, and a perplex'd Series of Conclusions, do deduce that universal Property of Proportionals which* Euclid *affirms; a Procedure foreign enough to the just Methods and Rules of Reasoning. They would certainly have done much better, if they had first laid down that universal*

Property assign'd by Euclid, *and thence have deduc'd that particular Property agreeing to only one Species of Proportionals. But rejecting this Method, they have taken the Liberty of adding their Demonstration to this Definition of the fifth Book. Those who have a mind to see a further Defence of* Euclid, *may consult the Mathematical Lectures of the learn'd Dr.* Barrow.

As I have happened to mention this great Geometrician, I must not pass by the Elements publish'd by him, wherein generally he has retain'd the Constructions and Demonstrations of Euclid *himself, not having omitted so much as one Proposition. Hence, his Demonstrations became more strong and nervous, his Construction more neat and elegant, and the Genius of the antient Geometricians more conspicuous, than is usually found in other Books of this kind. To this he has added, several Corollaries and Scholias, which serve not only to shorten the Demonstrations of what follows, but are likewise of use in other Matters.*

Not-

Notwithstanding this, Barrow's *Demonstrations are so very short, and are involv'd in so many Notes and Symboles; that they are render'd obscure and difficult to one not vers'd in Geometry. There are many Propositions which appear conspicuous in reading* Euclid *himself, are made knotty and scarcely intelligible to Learners by this Algebraical way of Demonstrations, as is, for Example,* Prop. 13. Book 1. *And the Demonstrations which he lays down in* Book 2. *are still more difficult:* Euclid *himself has done much better, in shewing their Evidence by the Contemplations of Figures, as in Geometry should always be done. The Elements of all Sciences ought to be handled after the most simple Method, and not to be involved in Symboles, Notes, or obscure Principles, taken elsewhere.*

As Barrow's *Elements are too short, so are those of* Clavius *too prolix, abounding in superfluous Scholiums and Comments: For in my Opinion,* Euclid *is not so obscure as to want such a number of Notes, neither do I doubt but a Learner will find* Euclid *himself, easier than any of his Com-*

mentators. As too much Brevity in Geometrical Demonstrations begets Obscurity, so too much Prolixity produces Tediousness and Confusion.

On these Accounts principally, it was that I undertook to publish the first six Books of Euclid, *with the 11th and 12th, according to* Commandinus's *Edition; the rest I forbore, because those first mention'd are sufficient for understanding of most parts of the Mathematicks now studied.*

Farther, for the Use of those who are desirous to apply the Elements of Geometry to Uses in Life, we have added a Compendium of Plain *aud* Spherical Trigonometry, *by means whereof Geometrical Magnitudes are measured, and their Dimensions expressed in Numbers.*

J. Keil.

Mr. *CUNN*'s PREFACE,

Shewing the Usefulness *and* Excellency *of this* WORK.

DR. *KEIL*, in his Preface, hath sufficiently declar'd how much easier, plainer, and eleganter, the Elements of Geometry written by *Euclid* are, than those written by others; and that the Elements themselves, are fitter for a Learner, than those

those publish'd by such as have pretended to Comment on, Symbolize, or Transpose any of his Demonstrations of such Propositions as they intended to treat of. Then how must a Geometrician be amaz'd, when he meets with a Tract * of the 1st, 2d, 3d, 4th, 5th, 6th, 11th, and 12th *Books* of the Elements, in which are omitted the Demonstrations of all the Propositions of that most noble universal Mathesis, the 5th; on which the 6th, 11th, and 12th so much depend, that the Demonstration of not so much as one Proposition in them can be obtain'd without those in the 5th.

The 7th, 8th, and 9th *Books* treat of such Properties of Numbers which are necessary for the Demonstrations of the 10th, which treats of Incommensurables; and the 13th, 14th, and 15th, of the five *Platonick* Bodies. But though the Doctrine of Incommensurables, because expounded in one and the same Plane, as the first six Elements were, claim'd by a Right of Order, to be handled before Planes intersected by Planes, or the more compounded Doctrine of Solids; and the Properties of Numbers were necessary to the Reasoning about Incommensurables:

Yet

* *Vide* the last Edition of the *English* Tacquet.

Yet because only one Proposition of these four Books, *viz.* the 1st of the 10th is quoted in the 11th and 12th Books; and that only once, *viz.* in the Demonstration of the 2d of the 12th, and that 1st Proposition of the 10th, is supplied by a *Lemma* in the 12th: And because the 7th, 8th, 9th, 10th, 13th, 14th, 15th Books have not been thought (by our greatest Masters) necessary to be read by such as design to make natural Philosophy their Study, or by such as would apply Geometry to practical Affairs, Dr. *Keil* in his Edition, gave us only these eight Books, *viz.* the first six, and the 11th and 12th.

And as he found there was wanting a Treatise of these Parts of the Elements, as they were written by *Euclid* himself; he publish'd his Edition without omitting any of *Euclid*'s Demonstrations, except two; one of which was a second Demonstration of the 9th Proposition of the third Book; the other a Demonstration of that Property of Proportionals call'd *Conversion*, (contain'd in a *Corollary* to the 19th Proposition of the 5th Book,) where instead of *Euclid*'s Demonstration, which is universal, most Authors have given us only particular ones of their own. The first of these which was omitted is here

sup-

ſupplied: And that which was corrupted is here reſtor'd *.

And ſince ſeveral Perſons to whom the Elements of Geometry are of vaſt Uſe, either are not ſo ſufficiently Skill'd in, or perhaps have not Leiſure, or are not willing to take the Trouble to read the *Latin*; and ſince this Treatiſe was not before in *Engliſh*, nor any other which may properly be ſaid to contain the Demonſtrations laid down by *Euclid* himſelf; I do not doubt but the Publication of this Edition will be acceptable, as well as ſerviceable.

Such Errors, either Typographical, or in the Schemes, which were taken Notice of in the *Latin* Edition, are corrected in this.

As to the Trigonometrical Tract annexed to theſe Elements, I find our Author, as well as Dr. *Harris*, Mr. *Caſwell*, Mr. *Heynes*, and others of the Trigonometrical Writers, is miſtaken in ſome of the Solutions.

That the common Solution of the 12th *Caſe* of Oblique Sphericks is falſe, I have demonſtrated, and given a true one. *See Page* 319.

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* *Vide Page* 55, 107, of *Euclid*'s Works, publiſh'd by Dr. *Gregory*.

In

In the Solution of our 9th and 10th *Cases*, by other Authors called the 1st and 2d, where are given and sought opposite Parts, not only the aforemention'd Authors, but all others that I have met with, have told us that the Solutions are ambiguous; which Doctrine is, indeed, sometimes true, but sometimes false: For sometimes the *Quæsitum* is doubtful, and sometimes not; and when it is not doubtful, it is sometimes greater than 90 Degrees, and sometimes less: And sure I shall commit no Crime, if I affirm, that no Solution can be given without a just Distinction of these Varieties. For the Solution of these *Cases* see my Directions at *Pages* 321, 322.

In the Solution of our 3d and 7th *Cases*, in other Authors reckon'd the 3d and 4th, where there are given two Sides and an Angle opposite to one of them, to find the 3d Side, or the Angle opposite to it; all the Writers of Trigonometry that I have met with, who have undertaken the Solutions of these two, as well as the two following *Cases*, by letting fall a Perpendicular, which is undoubtedly the shortest and best Method for finding either of these *Quæsita*, have told us, that the

Sum

$\left\{\begin{array}{l}\text{Sum}\\ \text{Difference}\end{array}\right\}$ of the Vertical Angles, or Bases, shall be the sought Angle or Side, according as the Perpendicular falls $\left\{\begin{array}{l}\text{within}\\ \text{without;}\end{array}\right\}$ which cannot be known, unless the Species of that unknown Angle, which is opposite to a given Side, be first known.

Here they leave us first to calculate that unknown Angle, before we shall know whether we are to take the Sum or the Difference of the Vertical Angles or Bases, for the sought Angle or Base: And in the Calculation of that Angle have left us in the dark as to its Species; as appears by my Observations on the two preceding *Cases*.

The Truth is the *Quæsitum* here, as well as in the two former *Cases*, is sometimes doubtful, and sometimes not; when doubtful, sometimes each Answer is less than 90 Degrees, sometimes each is greater; but sometimes one less, and the other greater, as in the two last mention'd *Cases*. When it is not doubtful, the *Quæsitum* is sometimes greater than 90 Degrees, and and sometimes less. All which Distinctions may be made without another Operation, or the Knowledge of the Species of that un-

unknown Angle, opposite to a given Side; or which is the same thing, the falling of the Perpendicular within or without. For which see my Directions at *Pages* 324, 325.

In the Solution of our 1st and 5th *Cases*, called in other Authors, the 5th and 6th; where there are given two Angles, and a Side opposite to one of them, to find the 3d Angle, or the Side opposite to it; they have told us, that the $\left\{\begin{matrix}\text{Sum}\\\text{Difference}\end{matrix}\right\}$ of the Vertical Angles, or Bases, according as the Perpendicular falls $\left\{\begin{matrix}\text{within}\\\text{without}\end{matrix}\right\}$ shall be the sought Angle or Side; and that it is known whether the Perpendicular falls within or without, by the Affection of the given Angles.

Here they seem to have spoken as tho' the *Quæsitum* was always determin'd, and never ambiguous; for they have here determined whether the Perpendicular falls within or without, and thereby whether they are to take the Sum or the Difference of the Vertical Angles or Bases, for the sought Angle or Side.

But, notwithstanding these imaginary Determinations, I affirm, that the *Quæ-*

situm here, as in the two *Cases* last mentioned, is sometimes ambiguous, and sometimes not; and that too, whether the Perpendicular falls within, or whether it falls without. See my Solutions of these two *Cases* in *Page* 323.

The Determination of the 3d *Case* of Oblique Plane Triangles. See in *Page* 325.

SAM. CUNN.

EUCLID's ELEMENTS.

BOOK I.

DEFINITIONS.

I. A *POINT, is that which hath no Parts, or Magnitude.*

II. *A Line is Length, without Breadth.*

III. *The Ends (or Bounds) of a Line, are Points.*

IV. *A Right Line, is that which lieth evenly between its Points.*

V. *A Superficies, is that which hath only Length and Breadth.*

VI. *The Bounds of a Superficies are Lines.*

VII. *A Plain Superficies, is that which lieth evenly between its Lines.*

VIII. *A Plain Angle, is the Inclination of two Lines to one another in the same Plane, which touch each other, but do not both lie in the same Right Line.*

IX. *If the Lines containing the Angle be Right ones, then the Angle is called a Right-lin'd Angle.*

X. *When a Right Line, standing on another Right Line, makes Angles on either Side thereof, equal between themselves, each of these equal Angles is a Right one; and that Right Line which stands upon the other, is called a Perpendicular to that whereon it stands.*

XI. *An Obtuse Angle, is that which is greater than a Right one.*

XII. *An Acute Angle, is that which is less than a Right one.*

XIII. *A Term (or Bound) is that which is the Extreme of any Thing.*

XIV. *A Figure, is that which is contained under one, or more Terms.*

XV. *A Circle, is a plain Figure, contain'd under one Line, called the Circumference; to which all Right Lines, drawn from a certain Point within the Figure, are equal.*

XVI. *And that Point is called the Center of the Circle.*

XVII. *A Diameter of a Circle, is a Right Line drawn through the Center, and terminated on both Sides by the Circumference, and divides the Circle into two equal Parts.*

XVIII. *A Semicircle, is a Figure contain'd under a Diameter, and that Part of the Circumference of a Circle cut off by that Diameter.*

XIX. *A Segment of a Circle, is a Figure contain'd under a Right Line, and Part of the Circumference of the Circle [which is cut off by that Right Line.]*

XX. *Right-lin'd Figures, are such as are contain'd under Right Lines.*

XXI. *Three-sided Figures, are such as are contained under three Lines.*

XXII. *Four-sided Figures, are such as are contain'd under four.*

XXIII. *Many sided Figures, are those that are contain'd under more than four Right Lines.*

XXIV. *Of three-sided Figures, that is an Equilateral Triangle, which hath three equal Sides.*

XXV. *That an Isosceles, or Equicrural one, which hath only two Sides equal.*

XXVI. *And a Scalene one, is that which hath three unequal Sides.*

XXVII. *Also of Three-sided Figures, that is, a Right-angled Triangle, which hath a Right Angle.*

XXVIII.

XXVIII. *That an Obtuse-angled one, which hath an Obtuse Angle.*

XXIX. *And that an Acute-angled one, which hath three Acute Angles.*

XXX. *Of Four-sided Figures, that is a Square, whose four Sides are equal, and its Angles all Right ones.*

XXXI. *That an Oblong, or Rectangle, a Figure which is longer on one side than the other, which is Right-angled, but not equal sided.*

XXXII. *That a Rhombus, which hath four equal Sides, but not Right Angles.*

XXXIII. *That a Rhomboides, whose opposite Sides and Angles only are equal.*

XXXIV. *All Quadrilateral Figures, besides these, are called Trapezia.*

XXXV. *Parallels are such Right Lines in the same Plane, which if infinitely produc'd both Ways, would never meet.*

POSTULATES.

I. GRANT *that a Right-Line may be drawn from any one Point to another.*

II. *That a finite Right Line may be continued directly forwards.*

III. *And that a Circle may be describ'd about any Center, with any Distance.*

AXIOMS.

I. THINGS *equal to one and the same Thing, are equal to one another.*

II. *If to equal Things, are added equal Things, the Wholes will be equal.*

III. *If from equal Things, equal Things be taken away, the Remainders will be equal.*

IV. *If equal Things be added to unequal Things, the Wholes will be unequal.*

V. *If equal Things be taken from unequal Things, the Remainders will be unequal.*

VI. *Things which are double to one and the same Thing, are equal between themselves.*

VII. *Things, which are half one and the same Thing, are equal between themselves.*

VIII. *Things which mutually agree together, are equal to one another.*

IX. *The Whole is greater than its Part.*

X. *Two Right Lines do not contain a Space.*

XI. *All Right Angles are equal between themselves.*

XII. *If a Right Line, falling upon two other Right Lines, makes the inward Angles on the same Side thereof, both together, less than two Right Angles, those two Right Lines, infinitely produc'd, will meet each other on that Side where the Angles, are less than Right ones.*

NOTE, When there are several Angles at one Point, any one of them is express'd by three Letters, of which that at the Vertex of the Angle is plac'd in the Middle. For Example; In the Figure of *Prop.* XII. *Lib.* I. the Angle contain'd under the Right Lines AB, BC, is called the Angle ABC; and the Angle contain'd under the Right Lines AB, BE, is call'd the Angle ABE.

PRO-

PROPOSITION I.

PROBLEM.

To describe an Equilateral Triangle upon a given finite Right Line.

LET AB be the given finite Right Line, upon which it is required to describe an Equilateral Triangle.

About the Center A, with the Distance AB, describe the Circle BCD *; and about the Center B, with the same Distance BA, describe the Circle ACE; and from the Point C, where the two Circles cut each other, draw the Right Lines CA, CB †. * *3 Post.* † *1 Post.*

Then because A is the Center of the Circle DBC, AC shall be equal to AB ‡. And because B is the Center of the Circle CAE, BC shall be equal to BA: but CA hath been proved to be equal to AB; therefore both CA and CB are each equal to AB. But things equal to one and the same thing, are equal between themselves, and consequently CA is equal to CB; therefore the three Sides CA, AB, BC, are equal between themselves. ‡ *15 Def.*

And so the Triangle BAC is an Equilateral one, and is described upon the given finite Right Line AB; which was to be done.

PROPOSITION II.

PROBLEM.

At a given Point, to put a Right Line equal to a Right Line given.

LET the Point given be A, and the given Right Line BC; it is required to put a Right Line at the Point A, equal to the given Right Line BC.

* *Post.* 1. Draw the Right Line AC from the Point A to C *,
† 1 *of this.* upon it describe the Equilateral Triangle DAC †;
‡ *Post.* 2. produce DA and DC directly forwards to E and C ‡; about the Center C, with the Distance BC, de-
* *Post.* 3. scribe the Circle BGH *; and about the Center D, with the Distance DG, describe the Circle GKL.

Now because the Point C is the Center of the Cir-
† *Def.* cle BGH, BC will be equal to CG †; and because D is the Center of the Circle GKL, the whole DL will be equal to the whole DG, the Parts whereof DA and DC are equal; therefore the Remainders AL,
‡ *Axiom* 3. GC are also equal ‡. But it has been demonstrated that BC is equal to CG; wherefore both AL and BC are each of them equal to CG. But Things that are equal to one and the same Thing, are equal to one another; and therefore likewise AL is equal to BC.

Whence the Right Line AL is put at the given Point A, equal to the given right Line BC, which was to be done.

PROPOSITION III.

PROBLEM.

Two unequal right Lines being given, to cut off a Part from the greater Equal to the lesser.

LET AB and C be the two unequal Right Lines given, the greater whereof is AB; it is required to cut off a Line from the greater AB equal to the lesser C.

* 2 *of this.* Put * a right Line AD at the Point A, equal to the Line C, and about the Center A, with the Distance
† *Post.* 3. AD, describe a Circle DEF †.

Then because A is the Center of the Circle DEF, AE is equal to AD; and so both AE and C are each
‡ *Axiom* 1. equal to AD; wherefore AE is likewise equal to C ‡.

And so there is cut off from AB the greater of two given Right Lines AB and C, a Line AE equal to the lesser Line C, which was to be done.

PROP.

PROPOSITION IV.

THEOREM.

If there are two Triangles that have two Sides of the one equal to two Sides of the other, each to each, and the Angle contained by those equal Sides in one Triangle equal to the Angle contained by the correspondent Sides in the other Triangle, then the Base of one of the Triangles shall be equal to the Base of the other, the whole Triangle equal to the whole Triangle, and the remaining Angles of one Equal to the remaining Angles of the other, each to each, which subtend the equal Sides.

LET the two Triangles be ABC, DEF, which have two Sides AB, AC, equal to two Sides DE, DF, each to each, that is, the Side AB equal to the Side DE, and the Side AC to DF; and the Angle BAC equal to the Angle EDF. I say, that the Base BC is equal to the Base EF, the Triangle ABC equal to the Triangle DEF, and the remaining Angles of the one equal to the remaining Angles of the other, each to its Correspondent, subtending the equal Sides, *viz.* the Angle ABC equal to the Angle DEF, and the Angle ACB equal to the Angle DFE.

For the Triangle ABC being applied to DEF, so as the Point A may co-incide with D, and the Right Line AB with DE, then the Point B will co-incide with the Point E, because AB is equal to DE. And since AB co-incides with DE, the Right Line AC likewise will co-incide with the Right Line DF, because the Angle BAC is equal to the Angle EDF. Wherefore also C will co-incide with F, because the Right Line AC is equal to the Right Line DF. But the Point B co-incides with E, and therefore the Base BC co-incides with the Base EF. For if the Point B co-inciding with E, and C with F, the Base BC does not co-incide with the Base EF; then two Right Lines will contain a Space, which is impossible*. Therefore the Base BC co-incides with the Base EF, and is equal thereto; and consequently the whole Triangle ABC will co-incide with the whole Triangle DEF, and

* *Ax.* 10.

and will be equal thereto; and the remaining Angles will co-incide with the remaining Angles †, and will be equal to them, *viz.* the Angle ABC equal to the Angle DEF, and the Angle ACB equal to the Angle DFE. Which was to be demonstrated.

† Ax. 8.

PROPOSITION V.

THEOREM.

The Angles at the Base of an Isosceles *Triangle are equal between themselves: And if the equal Sides be produced, the Angles under the Base shall be equal between themselves.*

LET ABC be an Isosceles Triangle, having the Side AB equal to the Side AC; and let the equal Sides AB, AC, be produced directly forwards to D and E. I say the Angle ABC is equal to the Angle ACB, and the Angle CBD equal to the Angle BCE.

For assume any Point F in the Line BD, and from AE cut off the Line AG equal * to AF, and join FC, GB.

* 3 *of this.*

Then because AF is equal to AG, and AB to AC, the two Right Lines FA, AC, are equal to the two Lines GA, AB, each to each, and contain the common Angle FAG; therefore the Base FC is equal † to the Base GB, and the Triangle AFC equal to the Triangle AGB, and the remaining Angles of the one equal to the remaining Angles of the other, each to each, subtending the equal Sides, *viz.* the Angle ACF equal to the Angle ABG; and the Angle AFC equal to the Angle AGB. And because the whole AF is equal to the whole AG, and the Part AB equal to the Part AC, the Remainder BF is equal to the Remainder CG. But FC ‡ has been proved to be equal to GB; therefore the two Sides BF, FC, are equal to the two Sides CG, GB, each to each, and the Angle BFC equal to the Angle CGB; but they have a common Base BC. Therefore also the Triangle BFC will be equal to the Triangle CGB, and the remaining Angles of the one equal to the remaining Angles of the other, each to each, which subtend the

† 4 *of this.*

‡ Ax. 3.

the equal Sides. And so the Angle FBC is equal to the Angle GCB; and the Angle BCF equal to the Angle CBG. Therefore because the whole Angle ABG has been proved equal to the whole Angle ACF, and the Part CBG equal to BCF, the remaining Angle ABC will be* equal to the remaining Angle ACB; but these are the Angles at the Base of the Triangle ABC. It hath likewise been proved that the Angles FBC, GCB, under the Base, are equal; therefore the Angles at the Base of Isosceles Triangles are equal between themselves; and if the equal Right Lines be produced, the Angles under the Base will be also equal between themselves. * *Ax.* 3.

Coroll. Hence every Equilateral Triangle is also Equiangular.

PROPOSITION VI.

THEOREM.

If two Angles of a Triangle be equal, then the Sides subtending the equal Angles will be equal between themselves.

LET ABC be a Triangle, having the Angle ABC equal to the Angle ACB. I say the Side AB is likewise equal to the Side AC.

For if AB be not equal to AC, let one of them, as AB, be the greater, from which cut off BD equal to AC*, and join DC. Then because DB is equal to AC, and BC is common, DB, BC, will be equal to AC, CB, each to each, and the Angle DBC equal to the Angle ACB, from the Hypothesis; therefore the Base DC is equal † to the Base AB, and the Triangle DBC equal to the Triangle ACB, a part to the whole, which is absurd; therefore AB is not unequal to AC, and consequently is equal to it. * 3 *of this.* † 4 *of this.*

Therefore if two Angles of a Triangle be equal between themselves, the Sides subtending the equal Angles are likewise equal between themselves. Which was to be demonstrated.

Coroll.

Coroll. Hence every Equiangular Triangle is alſo Equilateral.

PROPOSITION VII.

THEOREM.

On the ſame Right Line cannot be conſtituted two Right Lines equal to two other Right Lines, each to each, at different Points, on the ſame Side, and having the ſame Ends which the firſt Right Lines have.

FOR, if it be poſſible, let two Right Lines A D, D B, equal to two others A C, C B, each to each, be conſtituted at different Points C and D, towards the ſame Parts C D, and having the ſame Ends A and B which the firſt Right Lines have, ſo that C A be equal to A D, having the ſame End A which C A hath; and C B equal to D B, having the ſame End B; and let C D be joined.

Then becauſe A C is equal to A D, the Angle A C D will be equal* to the Angle A D C, and conſequently the Angle A D C is greater than the Angle B C D; wherefore the Angle B D C will be much greater than the Angle B C D. Again, becauſe C B is equal to D B, the Angle B D C will be equal to the Angle B C D; but it has been proved to be much greater, which is impoſſible. Therefore *on the ſame Right Line cannot be conſtituted two Right Lines equal to two other right Lines, each to each, at different Points, on the ſame Side, and having the ſame Ends which the firſt right Lines have*; which was to demonſtrated.

* *5 of this.*

PRO-

PROPOSITION VIII.

THEOREM.

If two Triangles have two Sides of the one equal to two Sides of the other, each to each, and the Bases equal, then the Angles contained under the equal Sides will be equal.

LET the two Triangles be ABC, DEF, having two Sides A B, AC, equal to two Sides D E, D F, each to each, *viz.* AB equal to D E, and AC to DF; and let the Base B C be equal to the Base EF. I say, the Angle BAC is equal to the Angle EDF.

For if the Triangle ABC be applied to the Triangle DEF, so that the Point B may co-incide with E, and the Right Line BC with EF, then the Point C will co-incide with F, because B C is equal to EF. And so since B C co-incides with E F, B A and AC will likewise co-incide with E D and D F. For if the Base B C should co-incide with E F, and at the same time the Sides BA, AC, should not co-incide with the Sides ED, DF, but change their Position, as EG, GF, then there would be constituted on the same Right Line two Right Lines equal to two other Right Lines, each to each, at several Points, on the same Side, having the same Ends. But this is proved to be otherwise *; therefore it is impossible for the Sides BA, AC, not to co-incide with the Sides ED, D F, if the Base B C co-incides with the Base E F; wherefore they will co-incide, and consequently the Angle B A C will co-incide with the Angle E D F, and will be equal to it. Therefore *if two Triangles have two Sides of the one equal to two Sides of the other, each to each, and the Bases equal, then the Angles contained under the equal Sides will be equal*; which was to be demonstrated.

* *7 of this.*

PRO-

PROPOSITION IX.

PROBLEM.

To cut a given Right-lin'd Angle into two equal Parts.

LET BAC be a given Right-lin'd Angle, which is required to be cut into two equal Parts.

Assume any Point D in the Right Line AB, and cut off AE from the Line AC equal to AD †; join DE, and thereon make ‡ the Equilateral Triangle DEF, and join AF. I say, the Angle BAC is cut into two equal Parts by the Line AF.

† 3 *of this.*
‡ 1 *of this.*

For because AD is equal to AE, and AF is common, the two Sides DA, AF, are each equal to the two Sides AE, AF, and the Base DF is equal to the Base EF; therefore * the Angle DAF is equal to the Angle EAF. Wherefore *a given Right-lin'd Angle is cut into two equal Parts*; which was to be done.

* 8 *of this.*

PROPOSITION. X.

PROBLEM.

To cut a given finite Right Line into two equal Parts.

LET AB be a given finite Right Line, required to be cut into two equal Parts.

Upon it make † an Equilateral Triangle ABC, and bisect ‡ the Angle ACB by the Right Line CD. I say, the right Line AB is bisected in the Point D.

† 1 *of this.*
‡ 9 *of this.*

For because AC is equal to CB, and CD is common, the Right Lines AC, CD, are each equal to the two Right Lines BC, CD, and the Angle ACD equal to the Angle BCD; therefore * the Base AD, is equal to the Base DB. And so the Right Line AB is bisected in the Point D; *which was to be done.*

* 4 *of this.*

PROPOSITION XI.

PROBLEM.

To draw a Right Line at Right Angles to a given Right Line, from a given Point in the same.

LET AB be the given Right Line, and C the given Point. It is required to draw a Right Line from the Point C, at Right Angles to A B.

Assume any Point D in A C, and make C E equal * to C D, and upon D E make † the Equilateral Triangle F D E, and join F C. I say, the Right Line F C is drawn from the Point C, given in the Right Line A B at Right Angles to A B. * 3 *of this.* † 1 *of this.*

For because D C is equal to C E, and F C is common, the two Lines D C, C F, are each equal to the two Lines E C, C F; and the Base D F is equal to the Base F E. Therefore the Angle D C F, is equal to the Angle E C F; and they are adjacent Angles. But when a Right Line, standing upon a Right Line, makes the adjacent Angles equal, each of the equal Angles is ‡ a Right Angle; and consequently D C F, F C E, are both Right Angles. Therefore the Right Line F C, &c. *which was to be done.* ‡ *Def.* 10.

PROPOSITION XII.

PROBLEM.

To draw a Right Line perpendicular, upon a given infinite Right Line, from a Point given out of it.

LET AB be the given infinite Line, and C the Point given out of it. It is requir'd to draw a Right Line perpendicular upon the given Right Line A B, from the Point C given out of it.

Assume any Point D on the other side of the Right Line A B, and about the Center C, with the Distance C D describe * a Circle E D G, bisect † E G in H, and join C G, C H, C E. I say there is drawn the Per- * *Post.* 3. † 10 *of this.*

Perpendicular CH on the given infinite Right Line AB, from the Point C given out of it.

For because GH is equal to HE, and HC is common, GH and HC are each equal to EH and HC, and the Base CG is equal to the Base CE. Therefore ‡ *Def.* 10. the Angle CHG is equal ‡ to the Angle CHE; and they are adjacent Angles. But when a Right Line, standing upon another Right Line, makes the Angles equal between themselves, each of the equal Angles * *Def.* 10. is a Right one*, and the said standing Right Line is call'd a Perpendicular to that which it stands on. Therefore GH is drawn *perpendicular*, *upon a given infinite Right Line*, *from a given Point out of it*; which was to be demonstrated.

PROPOSITION XIII.

THEOREM.

When a Right Line, standing upon a Right Line, makes Angles, these shall be either two Right Angles, or together equal to two Right Angles.

FOR let a Right Line AB, standing upon the Right Line CD, make the Angles CBA, ABD. I say, the Angles CBA, ABD, are either two Right Angles, or both together equal to two Right Angles.

* *Def.* 10. For if CBA be equal to ABD, they are * each of † 11 *of this.* them Right Angles: But if not, draw † BE from the Point B, at Right Angles to CD. Therefore the Angles CBE, EBD, are two Right Angles: And because CBE is equal to both the Angles CBA, ABE, add the Angle EBD, which is common; and the two ‡ *Ax.* 2. Angles CBE, EBD, together, are ‡ equal to the three Angles CBA, ABE, EBD, together. Again, because the Angle DBA is equal to the two Angles DBE, EBA, together, add the common Angle ABC, and the two Angles DBA, ABC, are equal to the three Angles DBE, EBA, ABC, together. But it has been prov'd that the two Angles CBE, EBD, together, are likewise equal to these three Angles: But Things that are equal to one and the same, * *Ax.* 1. are * equal between themselves. Therefore likewise the Angles CBE, EBD, together, are equal to the

 Angles

Angles DBA, ABC, together; but CBE, EBD, are two Right Angles. Therefore the Angles DBA, ABC, are both together equal to two Right Angles. *Wherefore when a Right Line, standing upon another Right Line, makes Angles, these shall be either two Right Angles, or together equal to two Right Angles;* which was to be demonstrated.

PROPOSITION XIV.

THEOREM.

If to any Right Line, and Point therein, two Right Lines be drawn from contrary Parts, making the adjacent Angles, both together, equal to two Right Angles, the said two Right Lines will make but one straight Line.

FOR let two Right Lines BC, BD, drawn from contrary Parts to the Point B, in any Right Line AB, make the adjacent Angles ABC, ABD, both together, equal to two Right Angles. I say, BC, BD, make but one Right Line.

For if BD, CB, do not make one straight Line, let CB and BE make one.

Then, because the Right Line AB stands upon the Right Line CBE, the Angles ABC, ABE, together, will be equal * to two Right Angles. But the Angles ABC, ABD, together, are also equal to two Right Angles. Now taking away the common Angle ABC, and the remaining Angle ABE is equal to the remaining Angle ABD, the less to the greater, which is impossible. Therefore BE, BC, are not one straight Line. And in the same Manner it is demonstrated, that no other Line but BD is in a straight Line with CB; wherefore CB, BD, shall be in one straight Line. Therefore *if to any Right Line, and Point therein two Right Lines be drawn from contrary Parts, making the adjacent Angles, both together, equal to two Right Angles, the said two Right Lines will make but one straight Line;* which was to be demonstrated.

* 13 of this.

PRO

PROPOSITION XV.

THEOREM.

If two Right Lines mutually cut each other, the opposite Angles are equal.

LET the two Right Lines AB CD mutually cut each other in the Point E. I say, the Angle AEC is equal to the Angle DEB; and the Angle CEB equal to the Angle AED.

For because the Right Line AE, standing on the Right Line CD, makes the Angles CEA, AED: These both together shall be equal * to two Right Angles. Again, because the Right Line DE standing upon the Right Line AB, makes the Angles AED, DEB: These Angles together are * equal to two Right Angles. But it has been prov'd, that the Angles CEA, AED, are likewise together equal to two Right Angles. Therefore the Angles CEA, AED, are equal to the Angles AED, DEB. Take away the Common Angle AED, and the Angle remaining CEA, is † equal to the Angle remaining BED. For the same Reason, the Angle CEB shall be equal to the Angle DEA. Therefore *if two Right Lines mutually cut each other, the opposite Angles are equal*; which was to be demonstrated.

* 13 *of this.*

† *Ax.* 3.

Coroll. 1. From hence it is manifest, that two Right Lines mutually cutting each other, make Angles at the Section equal to four Right Angles.

Coroll. 2. All the Angles constituted about the same Point, are equal to four Right Angles.

PRO-

PROPOSITION XVI.

THEOREM.

If one Side of any Triangle be produced, the outward Angle is greater than either of the inward Opposite Angles.

LET ABC be a Triangle, and one of its Sides BC, be produced to D. I say, the outward Angle ACD is greater than either of the inward Angles CBA, or BAC.

For bisect AC in E*, and join BE, which produce to F, and make EF equal to BE. Moreover, join FC, and produce AC to G. * 10 *of this.*

Then, because AE is equal to EC, and BE to EF, the two Sides AE, EB, are equal to the two Sides CE, EF, each to each, and the Angle AEB † equal to the Angle FEC; for they are opposite Angles. Therefore the Base AB, is ‡ equal to the Base FC; and the Triangle AEB, equal to the Triangle FEC; and the remaining Angles of the one, equal to the remaining Angles of the other, each to each, subtending the equal Sides. Wherefore the the Angle BAE, is equal to the Angle ECF; but the Angle ECD, is greater than the Angle ECF; therefore the Angle ACD, is greater than the Angle BAE. After the same manner, if the Right Line BC, be bisected, we demonstrate that the Angle BCG, that is, the Angle ACD, is greater than the Angle ABC. Therefore *one Side of any Triangle being produced, the outward Angle is greater than either of the inward opposite Angles*; which was to be demonstrated. † 15 *of this.* ‡ 4 *of this.*

PROPOSITION XVII.

THEOREM.

Two Angles of any Triangle together, howsoever taken, are less than two Right Angles.

LET ABC be a Triangle. I say, two Angles of it together, howsoever taken, are less than two Right Angles. C For

For produce BB to D.

Then because the outward Angle ACD of the Triangle ABC, is greater * than the inward opposite Angle ABC: If the common Angle ACB be added, the Angles ACD, ACB, together, will be greater than the Angles ABC, ACB together: But ACD, ACB, are † equal to two Right Angles. Therefore ABC, BCA, are less than two Right Angles. In the same manner we demonstrate that the Angles BAC, ACB, as also CAB, ABC, are less than two Right Angles. Therefore *two Angles of any Triangle together, howsoever taken, are less than two Right Angles*; which was to be demonstrated.

* 16 of this.
† 13 of this.

PROPOSITION. XVIII.

THEOREM.

The greater Side of every Triangle subtends the greater Angle.

LET ABC be a Triangle, having the Side AC greater than the Side AB. I say the Angle ABC is greater than the Angle BCA.

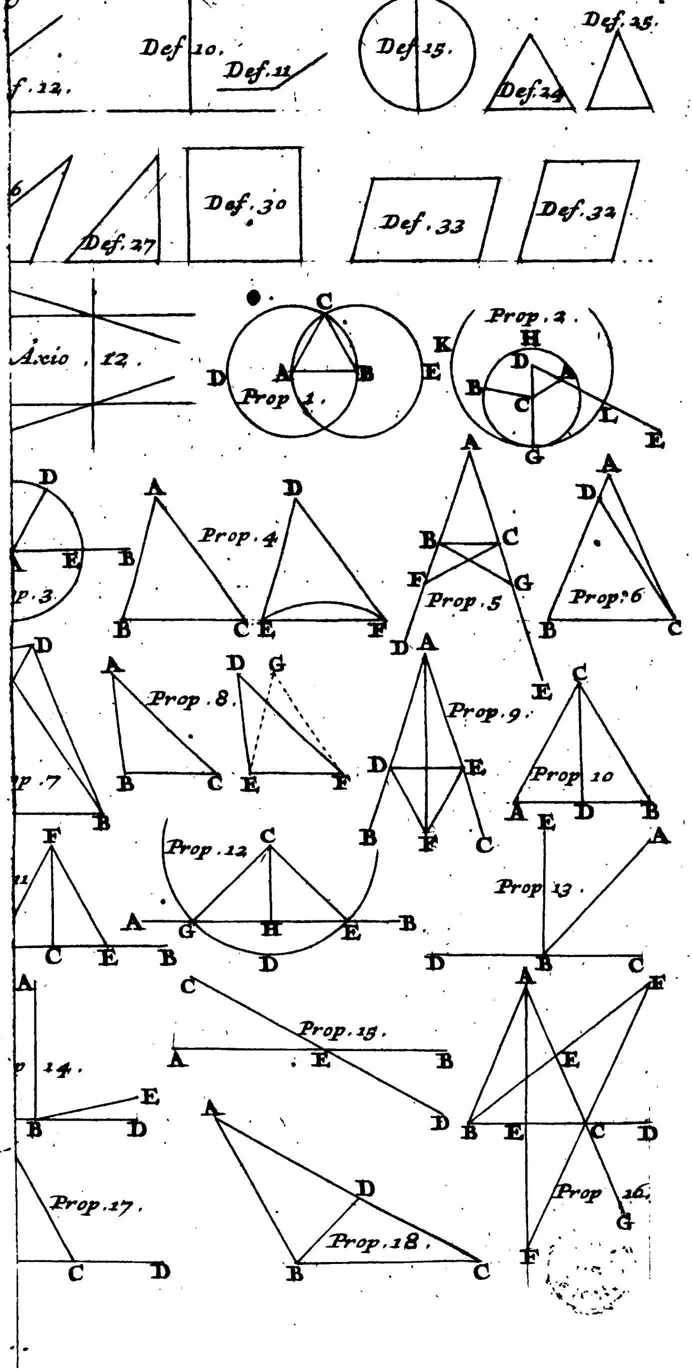
For because AC is greater than AB, AD may be made equal to AB, and BD be join'd.

Then because ADB is an outward Angle of the Triangle BDC, it will be * greater than the inward opposite Angle DCB. But ADB is † equal to ABD; because the Side AB is equal to the Side AD. Therefore the Angle ABD is likewise greater than the Angle ACB; and consequently ABC shall be much greater than ACB. Wherefore *the greater Side of every Triangle subtends the greater Angle*, which was to be demonstrated.

* 16 of this.
† 5 of this.

PRO-

Book 1. Plate 1.



PROPOSITION. XIX.

THEOREM.

The greater Angle of every Triangle subtends the greater Side.

LET ABC be a Triangle, having the Angle ABC greater than the Angle BCA. I say the Side AC is greater than the Side AB.

For if it be not greater, AC is either equal to AB, or less than it. It is not equal to it, because then the Angle ABC would be equal * to the Angle ACB; but it is not. Therefore AC is not equal to AB; neither will it be less; for then the Angle ABC would be † less than the Angle ACB; but it is not. Therefore AC is not less than AB. But likewise it has been prov'd not to be equal to it: Wherefore AC is greater than AB. Therefore *the greater Angle of every Triangle subtends the greater Side*; which was to be demonstrated.

* 5 *of this.*

† 18 *of this.*

PROPOSITION XX.

THEOREM.

Two Sides of any Triangle, howsoever taken, are together greater than the third Side.

LET ABC be a Triangle: I say two Sides thereof, howsoever taken, are together greater than the third Side; *viz.* the Sides BA, AC, are greater than the Side BC; and the Sides AB, BC, greater than the Side AC, and the Sides BC, CA, greater than the Side AB.

For produce BA to the Part D, so that AD be * equal to AC, and join DC.

Then because DA is equal to AC, the Angle ADC shall be equal † to the Angle ACD. But the Angle BCD is greater than the Angle ACD. Wherefore the Angle BCD is greater than the Angle ADC; and because DCB is a Triangle, having the Angle BCD greater than the Angle BDC, and the greater

* 3 *of this*

† 5 *of this*

*19 of this. Angle subtends * the greater Side; the Side DB will be greater than the Side BC. But DB is equal to BA and AC together. Wherefore the Sides BA, AC, together, are greater than the Side BC. In the same Manner we demonstrate, that the Sides AB, BC, together, are greater than the Side CA; and the Sides BC, CA, together, are greater than the Side AB. Therefore *two Sides of any Triangle, howsoever taken, are together greater than the third Side*; which was to be demonstrated.

PROPOSITION XXI.

THEOREM.

If two Right Lines be drawn from the extreme Points of one Side of a Triangle to any Point within the same, these two Lines shall be less than the other two Sides of the Triangle, but contain a greater Angle.

FOR let two Right Lines BD, DC, be drawn from the Extremes B, C, of the Side BC of the Triangle ABC, to the Point D within the same. I say BD, DC, are less than BA, AC, the other two Sides of the Triangle, but contain an Angle BDC greater than the Angle BAC.

For produce BD to E.

Then because two Sides of every Triangle together are * greater than the third, BA, AE, the two Sides of the Triangle ABE, are greater than the Side BE. Now add EC, which is common, and the Sides BA, AC, will be † greater than BE, EC.

*20 of this.

† Ax. 4.

Again, because CE, ED, the two Sides of the Triangle CED, are greater than the Side CD, add DB, which is common, and the Sides CE, EB, will be greater than CD, DB. But it has been prov'd, that BA, AC, are greater than BE, EC: Wherefore BA, AC, are much greater than BD, DC. Again, because the outward Angle of every Triangle, is ‡ greater than the inward and opposite one: BDC, the outward Angle of the Triangle CDE shall be greater than the Angle CED. For the same Reason CEB the outward Angle of the Triangle ABE, is likewise greater than the Angle BAC; but the Angle BDC has been

‡ 16 of this.

been prov'd to be greater than the Angle C E B. Wherefore the Angle B D C shall be much greater than the Angle B A C. And so *if two Right Lines be drawn from the extreme Points of one Side of a Triangle to any Point within the same; these two Lines shall be less than the other two Sides of the Triangle, but contain a greater Angle*; which was to be demonstrated.

PROPOSITION. XXII.

PROBLEM.

To describe a Triangle of three Right Lines which are equal to three others given: But it is requisite, that any two of the Right Lines taken together be greater than the third; because two Sides of a Triangle, howsoever taken, are together greater than the third Side.

LET A, B, C, be three Right Lines given, two of which, any ways taken, are greater than the third, *viz*. A and B together greater than C; A and C greater than B; and B and C greater than A. Now it is required to make a Triangle of three Right Lines equal to A, B, C: Let there be one Right Line D E terminated * at D, but infinite towards E; and take D F equal to A, F G equal to B, and G H equal to C; and about the Center F, with the Distance F D, describe a Circle D K L †; and about the Center G, with the Distance G H, describe another Circle K L H, and join K F, K G. I say, the Triangle K F G is made of three Right Lines equal to A, B, C, for because the Point F is the Center of the Circle D K; F K shall be equal to F D; but F D is equal to A; therefore F K is also equal to A. Again, because the Point G is the Center of the Circle L K H, G K will be ‡ equal to G H; but G H is equal to C: Therefore shall G K be also equal to C; but F G is likewise equal to B; and consequently the three Right Lines K F, F G, K G, are equal to the three Right Lines A, B, C; wherefore the Triangle K F G is made of three Right Lines K F, F G, G K, equal to the three given Lines A, B, C; *which was to be done.*

* 3 *of this.*

† 3 *Post.*

‡ *Def.* 15.

PROPOSITION XXIII.

PROBLEM.

With a given Right Line, and at a given Point in it, to make a Right-lin'd Angle equal to a Right-lin'd Angle given.

LET the given Right Line be AB, and the Point given therein A, and the given Right-lin'd Angle DCE. It is required to make a Right-lin'd Angle at the given Point A, with the given Right Line AB, equal to the given Right-lin'd Angle DCE.

Assume the Points D and E at Pleasure in the Lines CD, CE, and draw DE; then, of three Right Lines equal to CD, DE, EC, make * a Triangle AFG, so that AF be equal to CD, AG to CE, and FG to DE.

* 22 *of this.*

Then because the two Sides DC, CE, are equal to two Sides FA, AG, each to each, and the Base DE equal to the Base FG; the Angle DCE shall be † equal to the Angle FAG. Therefore the Right-lin'd Angle FAG is made, at the given Point A, in the given Line AB, equal to the given Right-lin'd Angle DCE; *which was to be done.*

† 8 *of this.*

PROPOSITION XXIV.

THEOREM.

If two Triangles have two Sides of the one, equal to two Sides of the other, each to each, and the Angle of the one, contained under the equal Right Lines, greater than the correspondent Angle of the other; then the Base of the one will be greater than the Base of the other.

LET there be two Triangles ABC, DEF, having two Sides AB, AC, equal to the two Sides DE, DF, each to each, *viz.* the Side AB equal to the Side DE, and the Side AC equal to DF; and let the Angle BAC be greater than the Angle EDF. I say, the Base BC is greater than the Base EF.

For

For because the Angle B A C is greater than the Angle E D F, make * an Angle E D G at the Point D in the Right Line D E, equal to the Angle B A C, and make † D G equal to either A C or D F, and join E F, F G. * 23 *of this.* † 3 *of this.*

Now because A B is equal to D E, and A C to D G, the two Sides B A, A C, are each equal to the two Sides E D, D G, and the Angle B A C equal to the Angle E D G: Therefore the Base B C is equal ‖ to the Base E G. Again, because D G is equal to D F, the Angle D F G is ‡ equal to the Angle D G F; and so the Angle D F G is greater than the Angle E G F: And consequently the Angle E F G is much greater than the Angle E G F. And because E F G is a Triangle, having the Angle E F G greater than the Angle E G F; and the greatest Side subtends ‡ the greatest Angle, the Side E G shall be greater than the Side E F. But the Side E G is equal to the Side B C. Whence B C is likewise greater than E F. Therefore *if two Triangles have two Sides of the one, equal to two Sides of the other, each to each, and the Angle of the one, contain'd under the equal Right Lines, greater than the correspondent Angle of the other; then the Base of the one will be greater than the Base of the other*; which was to be demonstrated. ‖ 4 *of this.* ‡ 5 *of this.* ‡ 19 *of this.*

PROPOSITION XXV.

THEOREM.

If two Triangles have two Sides of the one equal to two Sides of the other, each to each, and the Base of the one greater than the Base of the other; they shall also have the Angles, contain'd under the equal Sides, the one greater than the other.

LET there be two Triangles A B C, D E F, having two Sides A B, A C, each equal to two Sides D E, D F, *viz.* the Side A B equal to the Side D E, and the Side A C to the Side D F; but the Base B C greater than the Base E F. I say, the Angle B A C is also greater than the Angle E D F.

For if it be not greater, it will be either equal or less. But the Angle B A C is not equal to the Angle

* 4 of this. EDF; for if it was, the Base BC would be * equal to the Base EF; but it is not: Therefore the Angle BAC is not equal to the Angle EDF, neither will it † 24 of this. be lesser; for if it should, the Base BC would be † less than the Base EF; but it is not. Therefore the Angle BAC is not less than the Angle EDF; but it has likewise been prov'd not to be equal to it. Wherefore the Angle BAC is necessarily greater than the Angle EDF. *If, therefore, two Triangles have two Sides of the one equal to two Sides of the other, each to each, and the Base of the one greater than the Base of the other; they shall also have the Angles, contain'd under the equal Sides, the one greater than the other*; which was to be demonstrated.

PROPOSITION XXVI.

THEOREM.

If two Triangles have two Angles of the one equal to two Angles of the other, each to each, and one Side of the one equal to one Side of the other, either the Side lying between the equal Angles, or which subtends one of the equal Angles; the remaining Sides of the one Triangle shall be also equal to the remaining Sides of the other, each to his correspondent Side, and the remaining Angle of the one, equal to the remaining Angle of the other.

LET there be two Triangles ABC, DEF, having two Angles ABC, BCA of the one, equal to two Angles DEF, EFD, of the other, each to each, that is, the Angle ABC equal to the Angle DEF, and the Angle BCA equal to the Angle EFD. And let one Side of the one be equal to one Side of the other, which first let be the Side lying between the equal Angles, *viz.* the Side BC equal to the Side EF. I say, the remaining Sides of the one Triangle will be equal to the remaining Sides of the other, each to each, that is, the Side AB equal to the Side DE, and the Side AC equal to the Side DF, and the remaining Angle BAC equal to the remaining Angle EDF.

For

For if the Side AB be not equal to the Side DE, one of them will be the greater, which let be AB, make GB equal to DE, and join GC.

Then because BG is equal to DE, and BC to EF, the two Sides GB, BC, are equal to the two Sides DE, EF, each to each; and the Angle GBC equal to the Angle DEF. The Base GC is * equal to the Base DF, and the Triangle GBC to the Triangle DEF, and the remaining Angles equal to the remaining Angles, each to each, which subtend the equal Sides. Therefore the Angle GCB is equal to the Angle DFE. But the Angle DFE, by the Hypothesis, is equal to the Angle BCA; and so the Angle BCG is likewise equal to the Angle BCA, the less to the greater, which cannot be. Therefore AB is not unequal to DE, and consequently is equal to it. And so the two Sides AB, BC, are each equal to the two Sides DE, EF, and the Angle ABC equal to the Angle DEF: And consequently the Base AC * is equal to the Base DF, and the remaining Angle BAC equal to the remaining Angle EDF.

* *4 of this.*

Secondly, Let the Sides that are subtended by the equal Angles be equal, as AB equal to DE. I say, the remaining Sides of the one Triangle, are equal to the remaining Sides of the other, *viz.* AC to DF, and BC to EF; and also the remaining Angle BAC, to the remaining Angle EDF.

For if BC be unequal to EF, one of them is the greater, which let be BC, if possible, and make BH equal to EF, and join AH.

Now because BH is equal to EF, and AB to DE, the two Sides AB, BH, are equal to the two Sides DE, EF, each to each, and they contain equal Angles: Therefore the Base AH is * equal to the Base DF; and the Triangle ABH shall be equal to the Triangle DEF, and the remaining Angles equal to the remaining Angles, each to each, which subtend the equal Sides: And so the Angle BHA is equal to the Angle EFD. But EFD is † equal to the Angle BCA; and consequently the Angle BHA is equal to the Angle BCA: Therefore the outward Angle BHA of the Triangle AHC, is equal to the inward and opposite Angle BCA; which is ‡ impossible: Whence BC is not unequal to EF, therefore it is equal

† *From the Hyp.*

‡ *16 of this.*

equal to it. But AB is also equal to DE. Wherefore the two Sides AB, BC, are equal to the two Sides DE, EF, each to each; and they contain equal Angles. And so the Base AC is equal to the Base DF, the Triangle BAC to the Triangle DEF, and the remaining Angle BAC equal to the remaining Angle EDF. *If, therefore, two Triangles have two Angles equal, each to each, and one Side of the one equal to one Side of the other, either the Side lying between the equal Angles, or which subtends one of the equal Angles; the remaining Sides of the one Triangle shall be also equal to the remaining Sides of the other, each to his correspondent Side, and the remaining Angle of the one equal to the remaining Angle of the other;* which was to be demonstrated.

PROPOSITION XXVII.

THEOREM.

If a Right Line, falling upon two Right Lines, makes the alternate Angles equal between themselves, the two Right Lines shall be parallel.

LET the Right Line EF, falling upon two Right Lines AB, CD, make the alternate Angels AEF, EFD, equal between themselves. I say the Right Line AB is parallel to CD.

For if it be not parallel, AB and CD, produc'd towards B and D, or towards A and C, will meet: Now let them be produc'd towards B and D, and meet in the Point G.

Then the outward Angle AEF of the Triangle GEF, is * greater than the inward and opposite Angle EFG, and also equal † to it; which is absurd. Therefore AB and CD, produc'd towards B and D, will not meet each other. By the same Way of reasoning, neither will they meet, being produc'd towards C and A. But Lines that meet each other on neither Side, are ‡ parallel between themselves. Therefore AB is parallel to CD. Therefore, *if a Right Line, falling upon two Right Lines, makes the alternate Angles equal between themselves, the two Right Lines shall be parallel;* which was to be demonstrated.

* 16 of this.
† From the Hyp.
‡ Def. 35.

PROPOSITION XXVIII.

THEOREM.

If a Right Line, falling upon two Right Lines, makes the outward Angle of the one Line equal to the inward and opposite Angle of the other on the same Side, or the inward Angles on the same Side together equal to two Right Angles, the two Right Lines shall be parallel between themselves.

LET the Right Line EF falling upon two Right Lines AB, CD, make the outward Angle EGB equal to the inward and opposite Angle GHD; or the inward Angles BGH, GHD on the same Side together equal to two Right Angles. I say the Right Line AB is parallel to the Right Line CD.

For because the Angle EGB is * equal to the Angle GHD, and the Angle EGB † equal to the Angle AGH, the Angle AGH shall be equal to the Angle GHD; but these are alternate Angles. Therefore AB is ‡ parallel to CD. ** From the Hyp.* *† 15 of this.* *‡ 27 of this.*

Again, because the Angles BGH, GHD, are equal to two Right Angles, and AGH, BGH, are * equal to two Right Ones, the Angles AGH, BGH, will be equal to the Angles BGH, GHD; and if the common Angle BGH be taken from both, there will remain the Angle AGH equal to the Angle GHD; but these are alternate Angles. Therefore AB is parallel to CD. *If therefore a Right Line, falling upon two Right Lines, makes the outward Angle of the one Line equal to the inward and opposite Angle of the other on the same Side, or the inward Angles on the same Side together equal to two Right Angles, the two Right Lines shall be parallel between themselves*; which was to be demonstrated. ** 13 of this.*

PRO-

PROPOSITION XXIX.

THEOREM.

If a Right Line falls upon two Parallels, it will make the alternate Angles equal between themselves; the outward Angle equal to the inward and opposite Angle, on the same Side; and the inward Angles on the same Side together equal to two Right Angles.

LET the Right Line EF fall upon the parallel Right Lines AB, CD. I say the alternate Angles, AGH, GHD, are equal between themselves; the outward Angle, EGB, is equal to the inward one GHD, on the same Side; and the two inward ones, BGH, GHD, on the same Side, are together equal to two Right Angles.

For if AGH be unequal to GHD, one of them will be the greater. Let this be AGH; then because the Angle AGH is greater than the Angle GHD, add the common Angle BGH to both: And so the Angles AGH, BGH together, are greater than the Angles BGH, GHD, together. But the Angles AGH, BGH, are equal to two Right ones *. Therefore BGH, GHD, are less than two Right Angles. And so the Lines AB, CD, infinitely produc'd †, will meet each other; but because they are parallel, they will not meet. Therefore the Angle AGH is not unequal to the Angle GHD. Wherefore it is necessarily equal to it.

* 13 *of this.*

† *Ax.* 12.

But the Angle AGH is ‡ equal to the Angle EGB: Therefore EGB is also equal to GHD.

‡ 15 *of this.*

Now add the common Angle BGH, and then EGB, BGH, together, are equal to BGH, GHD, together; but EGB, and BGH, are equal to two Right Angles. Therefore also BGH, and GHD, shall be equal to two Right Angles. Wherefore, *if a Right Line falls upon two Parallels, it will make the alternate Angles equal between themselves; the outward Angle equal to the inward and opposite Angle, on the same Side, and the inward Angles on the same Side together equal to two Right Angles*; which was to be demonstrated.

PRO-

PROPOSITION XXX.

THEOREM.

Right Lines parallel to one and the same Right Line, are also parallel between themselves.

LET AB and CD be Right Lines, each of which is parallel to the Right Line EF. I say AB is also parallel to CD. For let the Right Line GK fall upon them.

Then because the Right Line GK falls upon the parallel Right Lines AB, EF, the Angle AGH is * equal to the Angle GHF; and because the Right Line, GK, falls upon the parallel Right Lines EF, CD, the Angle GHF is equal to the Angle GKD *. But it has been prov'd, that the Angle AGK is also equal to the Angle GHF. Therefore AGK is equal to GKD, and they are alternate Angles, whence AB is parallel to CD †. And so *Right Lines parallel to one and the same Right Line, are parallel between themselves*; which was to be demonstrated.

* 29 *of this.*

† 27 *of this.*

PROPOSITION XXXI.

PROBLEM.

To draw a Right Line thro' a given Point parallel to a given Right Line.

LET A be a Point given, and BC a Right Line given. It is requir'd to draw a Right Line thro' the Point A, parallel to the Right Line BC.

Assume any Point D in BC, and join AD; then make * an Angle DAE, at the Point A, with the Line DA, equal to the Angle ADC, and produce EA strait forwards to F.

Then because the Right Line AD falling on two Right Lines BC, EF, makes the alternate Angles EAD, ADC, equal between themselves, EF shall be † parallel to BC. Therefore the Right Line EAF is drawn thro' the given Point A, parallel to the given Right Line BC; *which was to be done.*

* 23 *of this.*

† 27 *of this.*

Coroll.

Coroll. Hence it appears, that if one Angle of any Triangle be equal to the other two, that is a Right one; because that the Angle adjacent to this Right one, is equal to the other two. But when adjacent Angles are equal, they are necessarily Right ones.

PROPOSITION XXXII.

THEOREM.

If one Side of any Triangle be produc'd, the outward Angle is equal to both the inward and opposite Angles; and the three inward Angles of a Triangle are equal to two Right Angles.

LET ABC be a Triangle, one of whose Sides BC is produc'd to D. I say, the outward Angle ACD is equal to the two inward and opposite Angles CAB, ABC; and the three inward Angles of the Triangle, *viz.* ABC, BCA, CAB, are equal to two Right Angles.

* 31 *of this.* For let CE be drawn * thro' the Point C parallel to the Right Line AB. Then because AB is parallel to CE, and AC falls upon them, the alternate Angles
† 29 *of this.* BAC, ACE, are † equal between themselves. Again, because AB is parallel to CE, and the Right Line BD falls upon them, the outward Angle ECD is † equal to the inward and opposite one ABC; but it has been prov'd, that the Angle ACE is equal to the Angle BAC. Wherefore the whole outward Angle ACD is equal to both the inward and opposite Angles BAC, ABC. And if the Angle ACB, which is common, be added, the two Angles ACD, ACB, are equal to the three Angles ABC, BAC, ACB; but the An-
‡ 13 *of this.* gles ACD, ACB, are ‡ equal to the two Right Angles. Therefore also shall the Angles ACB, CBA, CAB, be equal to two Right Angles. Wherefore *if one Side of any Triangle be produc'd, the outward Angle is equal to both the inward and opposite Angles, and the three inward Angles of a Triangle are equal to two Right Angles*; which was to be demonstrated.

Coroll.

Coroll. 1. All the three Angles of any one Triangle taken together, are equal to all the three Angles of any other Triangle taken together.

Coroll. 2. If two Angles of any one Triangle, either separately or taken together, be equal to two Angles of any other Triangle; then the remaining Angle of the one Triangle, will be equal to the remaining Angle of the other.

Coroll. 3. If one Angle of a Triangle be a Right Angle, the other two Angles together make one Right Angle.

Coroll. 4. If the Angle included between the equal Legs of an Isosceles Triangle be a Right one, each of the other Angles at the Base will be half Right Angles.

Coroll. 5. Any Angle in an Equilateral Triangle is equal to one Third of two Right Angles, or two Thirds of one Right Angle.

THEOREM I.

All the inward Angles of any Right-lin'd Figure whatsoever, make twice as many Right Angles, abating four, as the Figure has Sides.

FOR any Right-lin'd Figure may be resolv'd into as many Triangles, abating two, as it hath Sides. For Example, if a Figure has four Sides, it may be resolv'd into two Triangles: If a Figure has five Sides, it may be resolv'd into three Triangles; if six, into four; and so on. Wherefore (by Prop. XXXII.) *the Angles of all these Triangles are equal to twice as many Right Angles as there are Triangles: But the Angles of all the Triangles are equal to the inward Angles of the Figure: Therefore all the inward Angles of the Figure are equal to twice as many Right Angles as there are Triangles, that is, twice as many Right Angles, taking away four, as the Figure has Sides.* W. W. D.

THE-

THEOREM II.

All the outward Angles of any Right-lin'd Figure together, make four Right Angles.

FOR the outward Angles, together with the inward ones, make twice as many Right Angles as the Figure has Sides; but from the last Theorem, all the inward Angles together make twice as many Right Angles, abating four, as the Figure has Sides. Wherefore the outward Angles are all together equal to four Right Angles. W. W. D.

PROPOSITION. XXXIII.

THEOREM.

Two Right Lines, which join two equal and parallel Right Lines, towards the same Parts, are also equal and parallel.

LET the parallel and equal Right Lines A B, C D, be joined towards the same Parts, by the Right Lines A C, B D. I say A C, B D, are equal and parallel.

For draw B C.

Then because A B is parallel to C D, and B C falls upon them, the alternate Angles A B C, B C D, are * equal. Again, because A B is equal to C D, and B C is common; the two Sides A B, B C, are each equal to the two Sides B C, C D; but the Angle A B C is also equal to the Angle B C D; therefore the Base A C is † equal to the Base B D: And the Triangle A B C, equal to the Triangle B C D; and the remaining Angles equal to the remaining Angles, each to each, which subtend the equal Sides. Wherefore the Angle A C B is equal to the Angle C B D. And because the Right Line B C, falling upon two Right Lines A C, B D, makes ‡ the alternate Angles A C B, C B D, equal between themselves; A C is ‡ parallel to B D. But it has been prov'd also to be equal to it. Therefore *two Right Lines, which join two equal and parallel Right Lines, towards the same Parts, are also equal and parallel*; which was to be demonstrated.

* 29 *of this.*

† 4 *of this.*

‡ 27 *of this.*

Defin. *A Parallelogram is a Quadrilateral Figure, each of whose opposite Sides are parallel.*

PROPOSITION XXXIV.

THEOREM.

The opposite Sides and opposite Angles of any Parallelogram are equal; and the Diameter divides the same into two equal Parts.

LET ABDC be a Parallelogram, whose Diameter is BC. I say, the opposite Sides and opposite Angles are equal between themselves, and the Diameter BD bisects the Parallelogram.

For because AB is parallel to CD, and the Right Line BC falls on them, the alternate Angles ABC, BCD, are * equal between themselves. Wherefore ABC, CBD, are two Triangles, having two Angles ABC, BCA, of the one, equal to two Angles BCD, CBD, of the other, each to each; and likewise one Side of the one equal to one Side of the other, *viz.* the Side BC between the equal Angles, which is common. Therefore the remaining Sides shall be † equal to the remaining Sides, each to each, and the remaining Angle to the remaining Angle. And so the Side AB is equal to the Side CD, the Side AC to BD, and the Angle BAC to the Angle BDC. And because the Angle ABC is equal to the Angle BCD, and the Angle CBD to the Angle ACB; therefore the whole Angle ABD is equal to the whole Angle ACD: But it has been prov'd, that the Angle BAC is also equal to the Angle BDC. * 29 of this. † 26 of this.

Wherefore the opposite Sides and Angles of any Parallelogram are equal between themselves.

I say, moreover, that the Diameter bisects it. For because AB is equal to CD, and BC is common, the two Sides AB, BC, are each equal to the two Sides DC, CB; and the Angle ABC is also equal to the Angle BCD. Therefore the Base AC is ‡ equal to the Base DB; and the Triangle ABC is ‡ 4 of this.

equal to the Triangle BCD. Wherefore the Diameter BC bisects the Parallelogram ACDB; *which was to be demonstrated.*

PROPOSITION XXXV.

THEOREM.

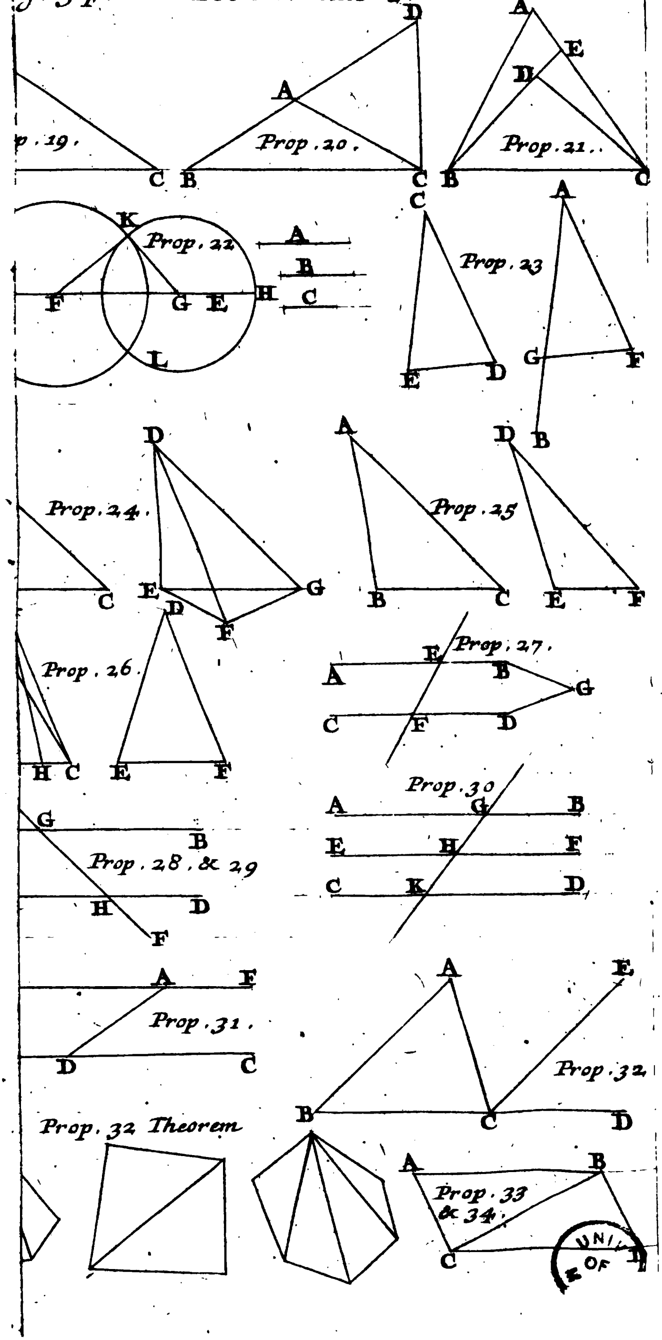
Parallelograms constituted upon the same Base, and between the same Parallels, are equal between themselves.

LET ABCD, EBCF, be Parallelograms constituted upon the same Base BC, and between the same Parallels AF and BC. I say, the Parallelogram ABCD, is equal to the Parallelogram EBCF.

For because ABCD is a Parallelogram, AD is
* *34 of this.* * equal to BC; and for the same Reason EF is equal
† *Axiom* 1. to BC; wherefore AD shall be † equal to EF; but
‡ *Ax.* 2. DE is common. Therefore the whole AE is ‡ equal to the whole DF. But AB is equal to DC; wherefore EA, AB, the two Sides of the Triangle ABE, are equal to the two Sides FD, DC, each to each;
* *29 of this.* and the Angle FDC * equal to the Angle EAB, the outward one to the inward one. Therefore the Base
† *4 of this.* EB is † equal to the Base EF, and the Triangle EAB to the Triangle FDC. If the common Triangle
‡ *Ax.* 3. DGE be taken from both, there will remain ‡ the Trapezium ABGD, equal to the Trapezium FCGE; and if the Triangle GBC which is common, be added, the Parallelogram ABCD will be equal to the Parallelogram EBCF. Therefore, *Parallelograms constituted upon the same Base, and between the same Parallels, are equal between themselves*; which was to be demonstrated.

PRO-

g. 34. Book 1. Plate 2.



PROPOSITION XXXVI.

THEOREM.

Parallelograms constituted upon equal Bases, and between the same Parallels, are equal between themselves.

LET the Parallelograms A B C D, E F G H, be constituted upon the equal Bases BC, FG, and between the same Parallels A H, B G. I say, the Parallelogram A B C D is equal to the Parallelogram EFGH.

For join BE, CH. Then because BC is * equal to FG, and FG to EH; BC will be likewise equal to EH; and they are parallel, and BE, CH, joins them. But two Right Lines joining Right Lines which are equal and parallel the same Way, are † equal, and parallel: Wherefore EBCH is a Parallelogram, and is ‡ equal to the Parallelogram ABCD; for it has the same Base BC, and is constituted between the same Parallels BC, AD. For the same Reason, the Parallelogram EFGH is equal to the same Parallelogram EBCH. Therefore the Parallelogram ABCD shall be equal to the Parallelogram EFGH. And so *Parallelograms constituted upon equal Bases, and between the same Parallels, are equal between themselves*; which was to be demonstrated.

* Hyp.

† 33 of this.

‡ 35 of this.

PROPOSITION XXXVII.

THEOREM.

Triangles constituted upon the same Base, and between the same Parallels, are equal between themselves.

LET the Triangles ABC, DBC, be constituted upon the same Base BC, and between the same Parallels AD, BC. I say, the Triangle ABC, is equal to the Triangle DBC.

For produce AD both ways to the Points E and F; and thro' B draw * BE parallel to CA; and thro' C, CF, parallel to BD.

* 31 of this.

Wherefore both EBCA, DBCF, are Parallelograms; and the Parallelogram EBCA is * equal to the Parallelogram DBCF; for they stand upon the same Base BC, and between the same Parallels BC, EF. But the Triangle ABC is † one half of the Parallelogram EBCA, because the Diameter AB bisects it; and the Triangle DBC is one half of the Parallelogram DBCF, for the Diameter DC bisects it. But Things that are the Halves of equal Things, are ‡ equal between themselves. Therefore the Triangle ABC, is equal to the Triangle DBC. Wherefore, *Triangles constituted upon the same Base, and between the same Parallels, are equal between themselves*; which was to be demonstrated.

* 35 of this.
† 34 of this.
‡ Ax. 7.

PROPOSITION XXXVIII.

THEOREM.

Triangles constituted upon equal Bases, and between the same Parallels, are equal between themselves.

LET the Triangles ABC, DCE, be constituted upon the equal Bases BC, CE, and between the same Parallels BE, AD. I say the Triangle ABC is equal to the Triangle DCE.

For produce AD both Ways to the Points G, H: thro' B draw * BG parallel to CA; and thro' E, EH, parallel to DC.

Wherefore both GBCA, DCEH, are Parallelograms, and the Parallelogram GBCA is † equal to the Parallelogram DCEH: For they stand upon equal Bases, BC, CE, and between the same Parallels BE, GH. But the Triangle ABC is ‡ one half of the Parallelogram GBCA, for the Diameter AB bisects it; and the Triangle DCE ‡ is one half of the Parallelogram DCEH, for the Diameter DE bisects it. But Things that are the Halves of equal Things, are * equal between themselves. Therefore the Triangle ABC, is equal to the Triangle DCB. Wherefore *Triangles constituted upon equal Bases, and between the same Parallels, are equal between themselves*; which was to be demonstrated.

* 31 of this.
† 36 of this.
‡ 34 of this.
* Ax. 7.

PRO-

PROPOSITION XXXIX.

THEOREM.

Equal Triangles constituted upon the same Base, on the same Side, are in the same Parallels.

LET ABC, DBC, be equal Triangles, constituted upon the same Base BC, on the same Side. I say they are between the same Parallels. For let AD be drawn. I say AD is parallel to BC.

For if it be not parallel, draw * the Right Line AE thro' the Point A, parallel to BC, and draw EC. * 31 *of this.*

Then the Triangle ABC, † is equal to the Triangle EBC; for it is upon the same Base BC, and between the same Parallels BC, AE. But the Triangle ABC is ‡ equal to the Triangle DBC. Therefore the Triangle DBC is also equal to the Triangle EBC; a less to a greater, which is impossible. Wherefore AE is not parallel to BC: And by the same Way of Reasoning we prove, that no other Line but AD is parallel to BC. Therefore AD is parallel to BC. Wherefore *equal Triangles constituted upon the same Base, on the same Side, are in the same Parallels*; which was to be demonstrated. † 37 *of this.* ‡ *From Hyp.*

PROPOSITION XL.

THEOREM.

Equal Triangles constituted upon equal Bases, on the same Side, are between the same Parallels.

LET ABC, CDE, be equal Triangles, constituted upon equal Bases BC, CE. I say they are between the same Parallels. For let AD be drawn. I say AD is parallel to BE.

For if it be not, let AF be drawn * thro' A, parallel to BE, and draw FE. * 31 *of this.*

Then the Triangle ABC is † equal to the Triangle FCE; for they are constituted upon equal Bases, and between the same Parallels BE, AF. But the Triangle ABC is equal to the Triangle DCE. There- † 38 *of this.*

fore the Triangle DCE shall be equal to the Triangle FCE, the greater to the less, which is impossible. Wherefore AF is not parallel to BE. And in this Manner we demonstrate, that no Right Line can be parallel to BE, but AD. Therefore AD is parallel to BE. And so *equal Triangles constituted upon equal Bases, on the same Side, are between the same Parallels*; which was to be demonstrated.

PROPOSITION XLI.

PROBLEM.

If a Parallelogram and a Triangle have the same Base, and are between the same Parallels, the Parallelogram will be double to the Triangle.

LET the Parallelogram ABCD, and the Triangle EBC, have the same Base, and be between the same Parallels, BC. AE. I say the Parallelogram ABCD is double the Triangle EBC.

For join AC.

* 37 *of this.* Now the Triangle ABC is * equal to the Triangle EBC; for they are both constituted upon the same Base BC, and between the same Parallels BC, AE.

† 34 *of this.* But the Parallelogram ABCD is † double the Triangle ABC, since the Diameter AC bisects it. Wherefore likewise it shall be double to the Triangle EBC. If, therefore, *a Parallelogram and Triangle have both the same Base, and are between the same Parallels, the Parallelogram will be double to the Triangle*; which was to be demonstrated.

PROPOSITION XLII.

PROBLEM.

To constitute a Parallelogram equal to a given Triangle, in an Angle equal to a given Right-lin'd Angle.

LET the given Triangle be ABC, and the Right-lin'd Angle given D. It is requir'd to constitute a Parallelogram equal to the given Triangle ABC, in a Right-lin'd Angle equal to D.

Bisect

Bisect * BC in E, join AE, and at the Point E, in the Right Line EC, constitute † an Angle CEF equal to D. Also draw ‡ AG thro' A, parallel to EC, and thro' C the Right Line CG parallel to FE. * 10 of this. † 23 of this. ‡ 31 of this.

Now FECG is a Parallelogram: And because BE is equal to EC, the Triangle ABE shall be * equal to the Triangle AEC; for they stand upon equal Bases BE, EC, and are between the same Parallels BC, AG. Wherefore the Triangle ABC is double to the Triangle AEC. But the Parallelogram FECG is also double to the Triangle AEC; for it has the same Base, and is between the same Parallels. Therefore the Parallelogram FECG, is equal to the Triangle ABC, and has the Angle CEF equal to the Angle D. Wherefore the Parallelogram FECG is constituted equal to the given Triangle ABC, in an Angle ECF equal to a given Angle D; *which was to be done*. * 38 of this.

PROPOSITION. XLIII.

THEOREM.

In every Parallelogram the Complements of the Parallelograms, that stand about the Diameter, are equal between themselves.

LET ABCD be a Parallelogram, whose Diameter is AB; and let FH, EG, be Parallelograms standing about the Diameter BD. Now AK, KC, are called the Complements of them: I say the Complement AK is equal to the Complement KC.

For since ABCD is a Parallelogram, and BD is the Diameter thereof, the Triangle ABD * is equal to the Triangle BDC. Again, because HKFD is a Parallelogram, whose Diameter is DK, the Triangle HDK shall * be equal to the Triangle DFK; and for the same Reason the Triangle KBG is equal to the Triangle KEB. But since the Triangle BEK is equal to the Triangle BGK, and the Triangle HDK to DFK; the Triangle BEK, together with the Triangle HDK, is equal to the Triangle BGK, together with the Triangle DFK. But the whole Triangle ABD is likewise equal to the whole Triangle * 34 of this.

BDC. Wherefore the Complement remaining, AK, will be equal to the remaining Complement KC. Therefore *in every Parallelogram the Complements of the Parallelograms, that stand about the Diameter, are equal between themselves*; which was to be done.

PROPOSITION XLIV.

PROBLEM.

To apply a Parallelogram to a given Right Line, equal to a given Triangle, in a given Right-lin'd Angle.

LET the Right Line given be AB, the given Triangle C, and the given Right-lin'd Angle D. It is requir'd to the given Right Line AB, to apply a Parallelogram equal to the given Triangle C.

In an Angle equal to D, make the Parallelogram BEFG equal to * the Triangle C; in the Angle EBG, equal to D. Place BE in a straight Line with AB, and produce FG to H, and thro' A let AH be drawn † parallel to either GB, or FE, and join HB.

* 42 of this.
† 31 of this.

Now because the Right Line HF falls on the Parallels AH, EF, the Angles AHF, HFE, are ‡ equal to two Right Angles. And so BHF, HFE, are less than two Right Angles; but Right Lines making less than two Right Angles, with a third Line being infinitely produc'd, will meet * each other. Wherefore HB, FE, produc'd, will meet each other; which let be in K, thro' which * draw KL parallel to EA, or FH, and produce AH, GB, to the Points L and M.

‡ 29 of this.
* Ax. 12.

Therefore HLKF is a Parallelogram, whose Diameter is HK; and AG, ME, are Parallelograms about HK; whereof LB, BF, are the Complements. Therefore LB is † equal to BF. But BF is also equal to the Triangle C. Wherefore likewise LB shall be equal to the Triangle C; and because the Angle GBE is ‡ equal to the Angle ABM, and also equal to the Angle D, the Angle AMB shall be equal to the Angle D. Therefore to the given Right Line AB is apply'd a Parallelogram, equal to the given Triangle C, in the Angle ABM, equal to the given Angle D; *which was to be done.*

† 43 of this.
‡ 15 of this.

PRO-

PROPOSITION XLV.

PROBLEM.

To make a Parallelogram equal to a given Right-lin'd Figure, in a given Right-lin'd Angle.

LET ABCD be the given Right-lin'd Figure, and E the Right-lin'd Angle given. It is requir'd to make a Parallelogram equal to the Right-lin'd Figure ABCD in an Angle equal to E.

Let DB be join'd, and make * the Parallelogram FH equal to the Triangle ADB, in an Angle HKF, equal to the given Angle E. * 42 of this.

Then to the Right Line GH apply † the Parallelogram GM, equal to the Triangle DBC, in an Angle GHM, equal to the Angle E. † 44 of this.

Now, because the Angle E is equal to HKF, or GHM, the Angle HKF shall be equal to GHM; add KHG to both; and the Angles HKF, KHG, are, together, equal to the Angles KHG, GHM. But HKF, KHG, are ‡, together, equal to two Right Angles. Wherefore, likewise, the Angles KHG, GHM, shall be equal to two Right Angles: And so at the given Point H in the Right Line GH, two Right Lines KH, HM, not drawn on the same Side, make the adjacent Angles, both together, equal to two Right Angles; and consequently KH, HM * make one straight Line. And because the Right Line HG falls upon the Parallels KM, FG, the alternate Angles MHG, HGF, are ‡ equal. And if HGL be added to both, the Angles MHG, HGL, together, are equal to the Angles HGF, HGL, together. But the Angles MHG, HGL, are * together equal to two Right Angles. Wherefore likewise the Angles HGF, HGL, are together equal to two Right Angles; and so FG, GL, make one straight Line. And since KF is equal and parallel to HG, as likewise HG to ML, KF shall be † equal and parallel to ML, and the Right Lines KM, FL, join them. Wherefore KM, FL, are ‡ equal and parallel. Therefore KFLM is a Parallelogram. But since the Triangle ABD is equal to the Parallelogram HF, and the

‡ 29 of this.

* 14 of this.

† 30 of this.

‡ 35 of this.

the Triangle DBC to the Parallelogram GM; then the whole Right-lin'd Figure ABCD will be equal to the whole Parallelogram KFLM. Therefore the Parallelogram KFLM is made equal to the given Right-lin'd Figue ABCD, in an Angle FKM, equal to the given Angle E; *which was to be done.*

Coroll. It is manifest, from what has been said, how to apply a Parallelogram to a given Right Line, equal to a given Right-lin'd Figure in a given Right-lin'd Angle.

PROPOSITION XLVI.

PROBLEM.

To describe a Square upon a given Right Line,

LET AB be the Right Line given, upon which it is requir'd to describe a Square.

* 11 of this. Draw * AC at Right Angles to AB from the Point † 3 of this. A given therein; make † AD equal to AB, and thro' ‡ 31 of this. the Point D draw ‡ DE parallel to AB; also thro' B draw BE parallel to AD.

* 34 of this. Then ADEB is a Parallelogram; and so AB * is equal to DE, and AD to BE. But BA is equal to AD. Therefore the four Sides BA, AD, DE, EB, are equal to each other.

And so the Paralellogram ADEB is equilateral; I say it is likewise equiangular. For because the Right Line AD falls upon the Parallels AB, DE, the Angles BAD, ADE, are † equal to two Right Angles. († 29 of this.) But BAD is a Right Angle: Wherefore ADE is also a Right Angle; but the opposite Sides and opposite Angles of Parallelograms are ‡ equal. (‡ 34 of this.) Therefore each of the opposite Angles ABE, BED, are Right Angles; and consequently ADBE is a Rectangle: But it has been prov'd to be equilateral. Therefore it is necessarily a Square, and is describ'd upon the Right Line AB; *which was to be done.*

Coroll. Hence every Parallelogram that has one Right Angle is a Rectangle.

PRO-

PROPOSITION XLVII.

THEOREM.

In any Right-angled Triangle, the Square describ'd upon the Side subtending the Right Angle, is equal to both the Squares describ'd upon the Sides containing the Right Angle.

LET ABC be a Right-angled Triangle, having the Right Angle BAC. I say the Square describ'd upon the Right Line BC, is equal to both the Squares describ'd upon the Sides BA, AC.

For describe * upon BC the Square BDEC, and on BA, AC, the Squares GB, HC, and thro' the Point A draw AL parallel to BD, or CE; and let AD, FC, be join'd. * *46 of this.*

Then because the Angles BAC, BAG, † are Right ones, two Right Lines AG, AC, at the given Point A, in the Right Line BA, being on contrary Sides thereof, make the adjacent Angles equal to two Right Angles. Therefore CA, AG, make ‡ one straight Line; by the same Reason AB, AH, make one straight Line. And since the Angle DBC is equal to the Angle FBA, for each of them is a Right one, add ABC, which is common, and the whole Angle DBA is * equal to the whole Angle FBC. And since the two Sides AB, BD, are equal to the two Sides FB, BC, each to each, and the Angle DBA equal to the Angle FBC; the Base AD will be † equal to the Base FC, and the Triangle ABD equal to the Triangle FBC: But the Parallelogram BL is ‡ double to the Triangle ABD; for they have the same Base DB, and are between the same Parallels BD, AL. The Square GB is ‡ also double to the Triangle FBC; for they have the same Base FB, and are in the same Parallels FB, GC. But Things that are the Doubles of equal Things are * equal to each other. Therefore the Parallelogram BL is equal to the Square GB. After the same Manner, AE, BK, being joined, we prove, that the Parallelogram CL is equal to the Square HC. Therefore the whole Square DBEC is equal to the two Squares GB, HC. But the Square DBEC is describ'd on the Right Line BC,

† *Def. 30.*
‡ *14 of this.*
* *Ax. 2.*
† *4 of this*
‡ *41 of this.*
* *Ax. 6.*

BC, and the Squares GB, HC, on BA, AC. Therefore the Square BE, describ'd on the Side BC, is equal to the Squares describ'd on the Sides BA, AC. Wherefore *in any Right-angl'd Triangle, the Square describ'd upon the Side subtending the Right Angle, is equal to both the Squares describ'd upon the Sides containing the Right Angle.*

PROPOSITION XLVIII.

THEOREM.

If a Square describ'd upon one Side of a Triangle be equal to the Squares describ'd upon the other two Sides of the said Triangle, then the Angle contain'd by these two other Sides is a Right Angle.

If the Square describ'd upon the Side BC of the Triangle ABC, be equal to the Squares describ'd upon the other two Sides of the Triangle BA, AC: I say the Angle BAC is a Right one.

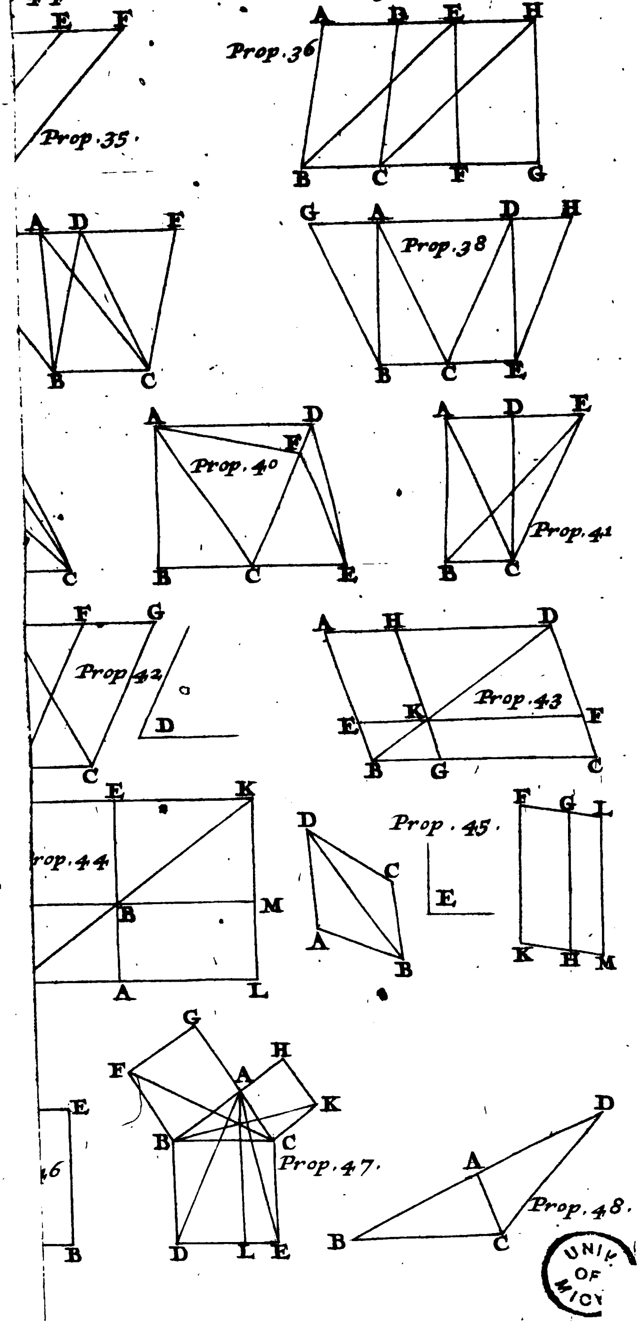
For let there be drawn AD from the Point A, at Right Angles to AC; likewise make AD equal to BA, and join DC.

Then because DA is equal to AB, the Square describ'd on DA will be equal to the Square describ'd on AB. And adding the common Square describ'd on AC, the Squares describ'd on DA, AC, are equal to the Squares describ'd on BA, AC. But the Square describ'd on DC is * equal to the Squares describ'd on DA, AC; for DAC is a Right Angle: But the Square on BC is put equal to the Squares on BA, AC. Therefore the Square describ'd on DC is equal to the Square describ'd on BC; and so the Side CD is equal to the Side CB. And because DA is equal to AB, and AC is common, the two Sides DA, AC, are equal to the two Sides BA, AC; and the Base DC is equal to the Base CB. Therefore the Angle DAC is † equal to the Angle BAC; but DAC is a Right Angle; and so BAC will be a Right Angle also. *If, therefore, a Square describ'd upon one Side of a Triangle be equal to the Squares describ'd upon the other two Sides of the said Triangle, then the Angle contain'd by these two other Sides is a Right Angle*; which was to be demonstrated.

* 47 of this.

† 8 of this.

Book 1 Plate 3.



EUCLID's ELEMENTS.

BOOK II.

DEFINITIONS.

I. *EVERY Right-angled Parallelogram is said to be contain'd under two Right Lines, comprehending a Right Angle.*

II. *In every Parallelogram, either of those Parallelograms that are about the Diameter, together with the Complements, is called a Gnomon.*

PRO-

PROPOSITION I.

THEOREM.

If there be two Right Lines, and one of them be divided into any Number of Parts; the Rectangle comprehended under the whole, and divided Line, shall be equal to all the Rectangles contained under the whole Line, and the several Segments of the divided Line.

LET A and BC be two Right Lines, whereof BC is cut or divided any how in the Points D, E. I say, the Rectangle contained under the Right Lines A and BC, is equal to the Rectangles contained under A and BD, A and DE, and A and EC.

* 11. 1. For let * BF be drawn from the Point B, at Right
† 3. 1. Angles, to BC; and make † BG equal to A; and let
‡ 31. 1. ‡ GH be drawn thro' G parallel to BC: Likewise, let ‡ there be drawn DK, EL, CH, thro' D, E, C, parallel to BG.

Then the Rectangle BH is equal to the Rectangles BK, DL, EH; but the Rectangle BH, is that contained under A and BC; for it is contain'd under GB, BC; and GB is equal to A; and the Rectangle BK is that contain'd under A and BD; for it is contain'd under GB and BD, and GB is equal to A; and the Rectangle DL is that contain'd under A and DE, because DK, that is, BG, is equal to A: So likewise the Rectangle EH is that contained under A and EC. Therefore the Rectangle under A and BC, is equal to the Rectangle under A and BD, A and DE, and A and EC. Therefore, *if there be two Right Lines given, and one of them be divided into any Number of Parts, the Rectangle comprehended under the whole and divided Line shall be equal to all the Rectangles contained under the whole Line, and the several Segments of the divided Line*; which was to be demonstrated.

PRO-

PROPOSITION II.

THEOREM.

If a Right Line be any how divided, the Rectangles contained under the whole Line, and each of the Segments, or Parts, are equal to the Square of the whole Line.

LET the Right Line AB be any how divided in the Point C. I say, the Rectangle contained under AB, BC, together with that contained under AB and AC, is equal to the Square made on AB.

For let the Square ADEB be described * on AB, and thro' C let CF be drawn parallel to AD or BE. Therefore AE is equal to the Rectangles AF and CE. But AE is a Square described upon AB; and AF is the Rectangle contain'd under BA, AC; for it is contained under DA and AC, whereof AD is equal to AB; and the Rectangle CE is contained under AB, BC, since BE is equal to AB. Wherefore the Rectangle under AB and AC, together with the Rectangle under AB and BC, is equal to the Square of AB. Therefore, *if a Right Line be any how divided, the Rectangles contained under the whole Line, and each of the Segments, or Parts, are equal to the Square of the whole Line.* * 46. 1.

PROPOSITION III.

THEOREM.

If a Right Line be any how cut, the Rectangle contained under the whole Line, and one of its Parts, is equal to the Rectangle contained under the two Parts together, with the Square of the first-mention'd Part.

LET the Right Line AB be any how cut in the Point C. I say, the Rectangle under AB and BC is equal to the Rectangle under AC and BC, together with the Square described on BC.

For describe * the Square CDEB upon BC; produce ED to F; and let AF be drawn † thro' A, Parallel to CD or BE. * 46. 1. † 31. 1.

2 Then

Then the Rectangle AE shall be equal to the two Rectangles AD, CE: And the Rectangle AE is that contained under AB and BC; for it is contained under AB and BE, whereof BE is equal to BC; And the Rectangle AD is that contained under AC and CB, since DC is equal to CB: And DB is a Square describ'd upon BC. Wherefore the Rectangle under AB and BC is equal to the Rectangle under AC and CB, together with the Square described upon BC. Therefore *if a Right Line be any how cut, the Rectangle contained under the whole Line, and one of its Parts, is equal to the Rectangle contained under the two Parts together, with the Square of the first-mention'd Part*; which was to be demonstrated.

PROPOSITION IV.

THEOREM.

If a Right Line be any how cut, the Square which is made on the whole Line will be equal to the Squares made on the Segments thereof, together with twice the Rectangle contained under the Segments.

LET the Right Line AB be any how cut in C. I say, the Square made on AB is equal to the Squares of AC, CB, together, with twice the Rectangle contained under AC, CB.

* 46. 1. For * describe the Square ADEB upon AB, join
† 41. 1. BD, and thro' C draw † CGF parallel to AD or BE; and also thro' G draw HK parallel to AB or DE.

Then because CF is parallel to AD, and BD falls
‡ 29. 1. upon them, the outward Angle BGC shall be ‡ equal to the inward and opposite Angle ADB: But the
* 5. 1. Angle ADB is * equal to the Angle ABD, since the Side BA is equal to the Side AD. Wherefore the Angle CGB is equal to the Angle GBC; and so the
† 6. 1. Side BC equal † to the Side CG; but likewise the
‡ 34. 1. Side CB is ‡ equal to the Side GK, and the Side CG to BK: Therefore GK is equal to KB, and CGKB is equilateral. I say, it is also Right-angled; for because CG is parallel to BK, and CB falls on them, the Angles KBC, GCB, are ‡ equal to two Right Angles. But KBC is a Right Angle. Wherefore

GBC also is a Right Angle, and the opposite Angles CGK, GKB, shall be Right Angles. Therefore CGKB is a Rectangle. But it has been proved to be equilateral. Therefore CGKB is a Square described upon BC. For the same Reason HF is also a Square made upon HG, that is equal to the Square of AC. Wherefore HF and CK are the Squares of AC and CB. And because the Rectangle AG is * equal to the Rectangle GE, and AG is that which is contained under AC and CB, for GC is equal to CB: GE shall be equal to the Rectangle under AC, and CB. Wherefore the Rectangles AG, GE, are equal to twice the Rectangle contained under AC, CB; and HF, CK, are the Squares of AC, CB. Therefore the four Figures HF, CK, AG, GE, are equal to the Squares of AC and CB, with twice the Rectangle contained under AC and CB. But HF, CK, AG, GE, make up the whole Square of AB, *viz.* ADEB. Therefore the Square of AB is equal to the Squares of AC, CB, together with twice the Rectangle contained under AC, CB. Wherefore, *if a Right Line be any how cut, the Square which is made on the whole Line, will be equal to the Squares made on the Segments thereof, together with twice the Rectangle contained under the Segments*; which was to be demonstrated. * 43 I.

Coroll. Hence it is manifest, that the Parallelograms which stand about the Diameter of a Square, are likewise Squares.

PROPOSITION V.

THEOREM.

If a Right Line be cut into two equal Parts, and into two unequal ones; the Rectangle under the unequal Parts, together with the Square that is made of the intermediate Distance, is equal to the Square made of half the Line.

LET any Right Line AB be cut into two equal Parts in C, and into two unequal Parts in D. I say the Rectangle contain'd under AD, DB, toge-

E ther

ther with the Square of CD, is equal to the Square of CB,

† 46. 1. For † describe C E F B, the Square of BC, draw BE, and thro' D draw * DHG, parallel to CE, or BF, and thro' H draw KLG, parallel to CB, or EF, and AK thro' A, parallel to CL, or BO.

* 31. 1.

‡ 43. 1. Now the Complement CH is ‡ equal to the Complement HF. Add DO, which is common to both of them, and the whole CO, is equal to the whole DF; but CO is † equal to AL, because AC is equal to CB; therefore AL is equal to DF, and adding CH, which is common, the whole AH shall be equal to FD, DL, together. But AH is the Rectangle contain'd under AD, DB; for DH is † equal to DB, * and FD, DL, is the Gnomon MNX; therefore MNX is equal to the Rectangle contain'd under AD, DB, and if LG, being common, and * equal to the Square of CD be added; then the Gnomon MNX, and LG, are equal to the Rectangle contain'd under AD, DB, together with the Square of CD; but the Gnomon MNX, and LG, make up the whole Square CEFB, *viz.* the Square of CB. Therefore the Rectangle under AD, DB, together with the Square of CD, is equal to the Square of CB. Wherefore, *if a Right Line be cut into two equal Parts, and into two unequal ones; the Rectangle under the unequal Parts, together with the Square that is made of the intermediate Distance, is equal to the Square made of half the Line*; which was to be demonstrated.

† 43. 1.

* Cor. 4. of this.

PRO-

PROPOSITION VI.

THEOREM.

If a Right Line be divided into two equal Parts, and another Right Line be added directly to the same, the Rectangle contain'd under the Line, compounded of the whole and added Line, (taken as one Line,) and the added Line, together with the Square of half the Line, is equal to the Square of the Line compounded of half the Line, and the added Line taken as one Line.

LET the Right Line AB be bisected in the Point C, and BD added directly thereto. I say the Rectangle under AD, and DB, together with the Square of BC, is equal the Square of CD.

For describe * CEFD, the Square of CD, and join DE; draw † BHG thro' B, parallel to CE, or DF, and KLM thro' H, parallel to AD, or EF, as also AK thro' A, parallel to CL, or DM. * 46. 1. † 36. 1.

Then because AC is equal to CB, the Rectangle AL shall ‡ be equal to the Rectangle CH, but CH is ‡ equal to HF. Therefore AL will be equal to HF; and adding CM, which is common to both, then the whole Rectangle AM, is equal to the Gnomon NXO. But AM is that Rectangle which is contain'd under AD, DB, for DM is * equal to DB; therefore the Gnomon NXO is equal to the Rectangle under AD, and DB. And adding LG, which is common, *viz.* † the Square of CB; and then the Rectangle under AD, DB, together with the Square of BC, is equal to the Gnomon NXO with LG. But the Gnomon NXO, and LG, together, make up the Figure CEFD, that is the Square of CD. Therefore the Rectangle under AD, and DB, together with the Square of BC, is equal to the Square of CD. Therefore, *if a Right Line be divided into two equal Parts, and another Right Line be added directly to the same, the Rectangle contain'd under the Line, compounded of the whole and added Line, (taken as one Line,) and the added Line, together with the Square* ‡ 43. 1. * Cor. 4. of this. † Cor. 4. of this.

of half the Line, is equal to the Square of the Line compounded of half the Line, and the added Line taken as one Line; which was to be demonstrated.

PROPOSITION VII.

THEOREM.

If a Right Line be any how cut, the Square of the whole Line, together with the Square of one of the Segments, is equal to double the Rectangle contain'd under the whole Line, and the said Segment, together with the Square, made of the other Segment.

LET the Right Line AB be any how cut in the Point C. I say the Squares of AB, BC, together, are equal to double the Rectangle contain'd under AB, BC, together with the Square, made of AC.

* 46. 1. For let the Square of AB be * describ'd, *viz.* ADEB, and construct ¶ the Figure.

† 43. 1. Then because the Rectangle AG, is † equal to the Rectangle GE. If CF, which is common, be added to both, the whole Rectangle AF shall be equal to the whole Rectangle CE, and so the Rectangles AF, CE, are double to the Rectangle AF; but AF, CE, make up the Gnomon KLM, and the Square CF. Therefore the Gnomon KLM, together with the Square CF, shall be double to the Rectangle AF. But double the Rectangle under AB, BC, is double the Rectangle AF, for BF is † equal to BC. Therefore the Gnomon KLM, and the Square CF, are equal to twice the Rectangle contain'd under AB, BC. And if HF, which is common, being the Square of AC, be added to both; then

---

¶ *A Figure is said to be constructed, when Lines, drawn in a Parallelogram parallel to the Sides thereof, cut the Diameter in one Point, and make two Parallelograms about the Diameter, and two Complements. So likewise a double Figure is said to be constructed, when two Right Lines parallel to the Sides, make four Parallelograms about the Diameter, and four Complements.*

then the Gnomon K L M, and the Squares C F, H F, are equal to double the Rectangle contain'd under A B, B C, together with the Square of A C. But the Gnomon K L M, together with the Squares C F, H F, are equal to A D E B, and C F, *viz.* the Squares of A B, B C. Therefore the Squares of A B, B C, are together equal to double the Rectangle contain'd under A B, B C, together with the Square of A C. Therefore, *if a Right Line be any how cut, the Square of the whole Line, together with the Square of one of the Segments, is equal to double the Rectangle contain'd under the whole Line, and the said Segment, together with the Square, made of the other Segment;* which was to be demonstrated.

PROPOSITION VIII.

THEOREM.

If a Right Line be any how cut into two Parts, four times the Rectangle, contain'd under the whole Line, and one of the Parts, together with the Square of the other Part, is equal to the Square of the Line, compounded of the whole Line, and the first Part taken as one Line.

LET the Right Line A B be cut any how in C. I say four times the Rectangle contain'd under A B, B C, together with the Square of A C, is equal to the Square of A B, and B C taken as one Line.

For let the Right Line A B be produced to D, so that B D be equal to B C, describe the Square A E F D, on A D, and construct the double Figure.

Now since C B is * equal to B D, and also to † G K, and B D is equal to K N: G K shall be likewise equal to K N; by the same Reasoning, P R is equal to R O. And since C B is equal to B D, and G K to K N, the Rectangle C K will ‡ be equal to the Rectangle B N, and the Rectangle G R to the Rectangle R N. But C K is * equal to R N; for they are the Complements of the Parallelogram C O. Therefore B N is equal to G R, and the four Squares B N, K C, G R, R N, are equal to each other;

* Hyp.
† 34. 1.
‡ 36. 1.
* 43 1.

 other;

other; and so they are together Quadruple C K. Again, because C B is equal to B D, and B D to B K, that is, equal to C G; and the said C B is equal also to G K, that is, to G P; therefore C G shall be equal to G P. But P R is equal to R O; therefore the Rectangle A G shall be equal to the Rectangle M P, and the Rectangle P L equal to R F. But M P is equal to P L; for they are the Complements of the Parallelogram M L. Wherefore A G is equal also to R F. Therefore the four Parallelograms A G, M P, P L, R F, are equal to each other, and accordingly they are together Quadruple of A G. But it has been prov'd that the four Squares C K, B N, G R, R N, are Quadruple of C K. Therefore the four Rectangles, and the four Squares, making up the Gnomon S T Y, are together Quadruple of A K; and because A K is a Rectangle contain'd under A B, and B C, for B K is equal to B C; four times the Rectangle under A B, B C, will be Quadruple of A K. But the Gnomon S T Y has been prov'd to be Quadruple of A K. And so four times the Rectangle contain'd under A B, B C, is equal to the Gnomon S T Y. And if X H, being equal to † the Square of A C, which is common, be added to both: Then four times the Rectangle contain'd under A B, B C, together with the Square of A C, is equal to the Gnomon S T Y, and the Square X H. But the Gnomon S T Y and H X, make A E F D, the whole Square of A D. Therefore four times the Rectangle contain'd under A B, B C, together with the Square of A C, is equal to the Square of A D, that is, of A B and B C taken as one Line. Wherefore, *if a Right Line be any how cut into two Parts, four times the Rectangle contain'd under the whole Line, and one of the Parts, together with the Square of the other Part, is equal to the Square of the Line, compounded of the whole Line, and the first Part taken as one Line*; which was to be demonstrated.

† *Cor.* 4. *of this.*

PRO-

PROPOSITION IX.

THEOREM

If a Right Line be any how cut into two equal, and two unequal Parts; then the Squares of the unequal Parts together, are double to the Square of the half Line, and the Square of the intermediate Part.

LET any Right Line AB be cut unequally in D, and equally in C. I say the Squares of AD, DB, together, are double to the Squares of AC and CD together.

For let * CE be drawn from the Point C at Right Angles to AB, which make equal to AC, or CB, and join EA, EB. Also thro' D let † DF be drawn parallel to CE, and FG thro' F parallel to AB, and draw AF. * 11. 1. † 31. 1.

Now because AC is equal to CE, the Angle EAC will be ‡ equal to the Angle AEC; and since the Angle at C is a Right one, the other Angles AEC, EAC, together, shall * make one Right Angle, and are equal to each other: And so AEC, EAC, are each equal to half a Right Angle. For the same Reason are also CEB, EBC, each of them half Right Angles. Therefore the whole Angle AEB is a Right Angle. And since the Angle GEF is half a Right one, and EGF is a Right Angle; for it is † equal to the inward and opposite Angle EGB, the other Angle EFG will be also equal to half a Right one. Therefore the Angle GEF is equal to the Angle EFG. And so the Side EG is ‡ equal to the Side GF. Again, because the Angle at B is half a Right one, and FDB is a Right one, because equal to the inward and opposite Angle ECB, the other Angle BFD will be half a Right Angle. Therefore the Angle at B is equal to the Angle BFD; and so the Side DF is equal to the Side DB. And because AC is equal to CE, the Square of AC will be equal to the Square of CE. Therefore the Squares of AC, CE, together, are double to the Square of AC; but the Square of EA is ⸸ equal to the Squares of AC, CE, together, since ACE is a Right Angle. Therefore the Square of ‡ 5. 1. * 3 Cor. 32. 1. † 29. 1. ‡ 6. 1. ⸸ 47. 1.

EA is double to the Square of AC. Again, because EG is equal to GF, and the Square of EG is equal to the Square of EF. Therefore the Squares of EG, GF, together, are double to the Square of GF. But the Square of EF is † equal to the Squares of EG, GF. Therefore the Square of EF is double the Square of GF: But GF is equal to CD; and so the Square of EF double to the Square of CD. But the Square of AE is likewise double to the Square of AC. Wherefore the Squares of AE, and EF, are double to the Squares of AC and CD. But the Square of AF is † equal to the Squares of AE and EF; because the Angle AEF is a Right Angle, and consequently the Square of AF is double to the Squares of AC, and CD. But the Squares of AD, DF, are equal to the Square of AF: For the Angle at D is a Right Angle. Therefore the Squares of AD, and DF, together, shall be double to the Squares of AC and CD together. But DF is equal to DB. Therefore the Squares of AD, and DB, together, will be double to the Squares of AC and CD, together. Wherefore, *if a Right Line be any how cut into two equal, and two unequal Parts, then the Squares of the unequal Parts together, are double to the Square of the half Line, and the Square of the intermediate Part*; which was to be demonstrated

† 47. 1.

PROPOSITION X.

THEOREM.

If a Right Line be cut into two equal Parts, and to it be directly added another; the Square made on [the Line compounded of] the whole Line, and the added one, together with the Square of the added Line, shall be double to the Square of the half Line, and the Square of [that Line which is compounded of] the half, and the added Line.

LET the Right Line AB be bisected in C, and any straight Line BD added directly thereto. I say the Squares of AD, DB, together, are double to the Squares of AC, CD, together.

For

For draw * CE from the Point C at Right Angles to AB, which make equal to AC, or CB, and draw AE, EB; likewise thro' E let EF be † drawn parallel to AD, and thro' D, DF † parallel to CE. * 11. 1. † 31. 1.

Then because the Right Line EF falls upon the Parallels EC, FD, the Angles CEF, EFD, are ‡ equal to two Right Angles. Therefore the Angles FEB, EFD, are together less than two Right Angles. But Right Lines making, with a third Line, Angles together less than two Right Angles, being infinitely produc'd, will meet *. Wherefore EB, FD, produc'd, will meet towards BD. Now let them be produc'd, and meet each other in the Point G, and let AG be drawn. ‡ 29. 1. * Ax. 12.

And then because AC is equal to CE, the Angle AEC will be equal to the Angle EAC †: But the Angle at C is a Right Angle. Therefore the Angle CAE, or AEC, is half a Right one. By the same way of Reasoning, the Angle CEB, or EBC, is half a Right one. Therefore AEB is a Right Angle. And since EBC is half a Right Angle, DBG will ‡ also be half a Right Angle, since it is vertical to CBE. But BDG is a Right Angle also; for it is ‡ equal to the alternate Angle DCE. Therefore the remaining Angle DGB is half a Right Angle, and so equal to DBG. Wherefore the Side BD is * equal to the Side DG. Again, because EGF is half a Right Angle, and the Angle at F is a Right Angle, for it is equal to the opposite Angle at C; the remaining Angle FEG will be also half a Right one, and is equal to the Angle EGF; and so the Side GF is * equal to the Side EF. And since EC is equal to CA, and the Square of EC equal to the Square of CA; therefore the Squares of EC, CA, together, are double to the Square of CA. But the Square of EA is † equal to the Squares of EC, CA. Wherefore the Square of EA is double to the Square of AC. Again, because GF is equal to FE, the Square of GF also is equal to the Square of FE. Wherefore the Squares of GF, FE, are double to the Square of FE. But the Square of EG is † equal to the Squares of GF, FE. Therefore the Square of EG is double to the Square of EF: But EF is equal to CD. Wherefore the Square of EG shall be double to the Square of CD. † 5. 1. ‡ 15. 1. * 6. 1. † 47. 1.

But

But the Square of EA has been prov'd to be double to the Square of AC. Therefore the Squares of AE, EG, are double the Squares of AC, CD. But

† 47. 1. the Squares of AG is † equal to the Squares of AE, EG; and consequently the Square of AG is double to the Squares of AC, CD. But the Squares of AD, DG, are † equal to the Square AG. Therefore the Squares of AD, DG, are double the Squares of AC, CD. But DG is equal to DB. Wherefore the Squares of AD, DB, are double to the Squares of AC, CD. Therefore *if a Right Line be cut into two equal Parts, and to it be directly added another; the Square made on [the Line compounded of] the whole Line, and the added one, together with the Square of the added Line, shall be double to the Square of the half Line, and the Square of [that Line which is compounded of] the half, and the added Line.*

PROPOSITION. XI.

PROBLEM.

To cut a given Right Line so, that the Rectangle contained under the whole Line, and one Segment, be equal to the Square of the other Segment.

LET AB be a given Right Line. It is required to cut the same so, that the Rectangle contained under the whole, and one Segment thereof, be equal to the Square of the other Segment.

* 46. 1. Describe * ABCD the Square of AB, bisect AC in E, and draw BE: Also, produce CA to F, so that EF be equal to EB. Describe FGHA the Square of AF, and produce GH to K. I say, AB is cut in H so, that the Rectangle under AB, BH, is equal to the Square of AH.

For since the Right Line AC is bisected in E, and AE is directly added thereto, the Rectangle under CF, FA, together with the Square of AE, will be

† 6 of this. † equal to the Square of EF. But EF is equal to EB. Therefore the Rectangle under CF, FA, together with the Square of AE, is equal to the Square

‡ 47 of this. of EB. But the Squares of BA, AE, are ‡ equal to the Square of EB; for the Angle at A is a Right Angle.

gle. Therefore the Rectangle under CF, FA, together with the Square of AE, is equal to the Squares of BA, AE. And taking away the Square of AE, which is common, the remaining Rectangle under CF, FA, is equal to the Square of AB. But FK is the Rectangle under CF, FA; since AF is equal to FG; and the Square of AB is AD. Wherefore the Rectangle FK is equal to the Square AD. And if AK, which is common, be taken from both, then the remaining Square FH is equal to the remaining Rectangle HD. But HD is the Rectangle under AB, BH, since AB is equal to BD, and FH is the Square of AH. Therefore the Rectangle under AB, BH shall be equal to the Square of AH: And so the given Right Line AB is cut in H, so that the Rectangle under AB, BH, is equal to the Square of AH. *Which was to be done.*

PROPOSITION XII.

THEOREM.

In obtuse angled Triangles, the Square of the Side subtending the obtuse Angle, is greater than the Squares of the Sides containing the obtuse Angle, by twice the Rectangle under one of the Sides, containing the obtuse Angle, viz. *that on which, produc'd, the Perpendicular falls, and the Line taken without, between the perpendicular and the obtuse Angle.*

LET ABC be an obtuse angled Triangle, having the obtuse Angle BAC; and * from the Point B draw BD perpendicular to the Side CA produc'd. I say the Square of BC is greater than the Squares of BA and AC, by twice the Rectangle contain'd under CA, and AD. * 12. 1.

For because the Right Line CD is any how cut in the Point A, the Square of CD shall be † equal to the Squares of CA, and AD, together with twice the Rectangle under CA, and AD. And if the Square of BD, which is common, be added, then the Squares of CD, DB, are equal to the Squares of CA, AD, and DB, and twice the Rectangle contain'd under † 4 *of this.*

CA.

* 47 1. CA and AD. But the Square of CB is * equal to the Squares of CD, DB; for the Angle at D is a Right one, since BD is perpendicular, and the Square of AB is * equal to the Squares of AD, and DB. Therefore the Square of CB is equal to the Squares of CA and AB, together with twice a Rectangle under CA and AD. Therefore *in obtuse angled Triangles, the Square of the Side subtending the obtuse Angle, is greater than the Squares of the Sides containing the obtuse Angle, by twice the Rectangle under one of the Sides containing the obtuse Angle,* viz. *that on which, produc'd, the Perpendicular falls, and the Line taken without, between the perpendicular and the obtuse Angle*; which was to be demonstrated.

PROPOSITION XIII.

THEOREM.

In acute angled Triangles, the Square of the Side subtending the acute Angles, is less than the Squares of the Sides containing the acute Angle, by twice a Rectangle under one of the Sides about the acute Angle, viz. *on which the Perpendicular falls, and the Line assum'd within the Triangle, from the Perpendicular to the acute Angle.*

LET ABC be an acute angled Triangle, having the acute Angle B: And from A let there * be drawn AD perpendicular to BC. I say the Square of AC is less than the Squares of CB and BA by twice a Rectangle under CB and BD.

† 7 of this. For because the Right Line CB is cut any how in D, the Squares of CB and BD will be † equal to twice a Rectangle under CB and BD, together with the Square of DC. And if the Square of AD be added to both, then the Squares of CB, BD, and DA, are equal to twice the Rectangle contain'd under CB and BD, together with the Squares of AD and DC. But the Square of AB is ‡ equal to the Squares of BD and DA; for the Angle at D is a Right Angle. And the Square of AC is ‡ equal to the Squares of AD and DC. Therefore the Squares of CB and BA are equal to the Square of AC, together

‡ 47. 1.

gether, with twice the Rectangle contain'd under CB and BD. Wherefore the Square of AC only, is less than the Squares of CB and BA, by twice the Rectangle under CB and BD. Therefore *in acute angled Triangles, the Square of the Side subtending the acute Angles, is less than the Squares of the Sides containing the acute Angle, by twice a Rectangle under one of the Sides about the acute Angle*, viz. *on which the Perpendicular falls, and the Line assum'd within the Triangle, from the perpendicular to the acute Angle*; which was to be demonstrated.

PROPOSITION XIV.

PROBLEM.

To make a Square equal to a given Right-lin'd Figure.

LET A be the given Right-lin'd Figure. It is requir'd to make a Square equal thereto.

Make * the Right-angled Parallelogram BCDE equal to the Right-lin'd Figure A. Now if BE be equal to ED, what was propos'd will be already done, since the Square BD is made equal to the Right-lin'd Figure A: But if it be not, let either BE or ED be the greater: Suppose BE, which let be produc'd to F; so that EF be equal to ED. This being done, let BE be † bisected in G, about which, as a Center, with the Distance GB or GF, describe the Semicircle BHF; and let DE be produc'd to H, and draw GH. Now because the Right Line BF is divided into two equal Parts in G, and into two unequal ones in E, the Rectangle under BE and EF, together with the Square of GE, shall be ‡ equal to the Square of GF. But GF is equal to GH. Therefore the Rectangle under BE, EF, together, with the Square of GE, is equal to the Square of GH. But the Squares of HE and EG are * equal to the Square of GH. Wherefore the Rectangle under BE, EF, together with the Square of EG, is equal to the Squares of HE, EG. And if the Square of EG, which is common, be taken from both, the remaining Rectangle contain'd under BE and EF, is equal to the Square of EH. But the Rectangle under BE and EF is the Parallelogram BD, because EF is equal to ED.

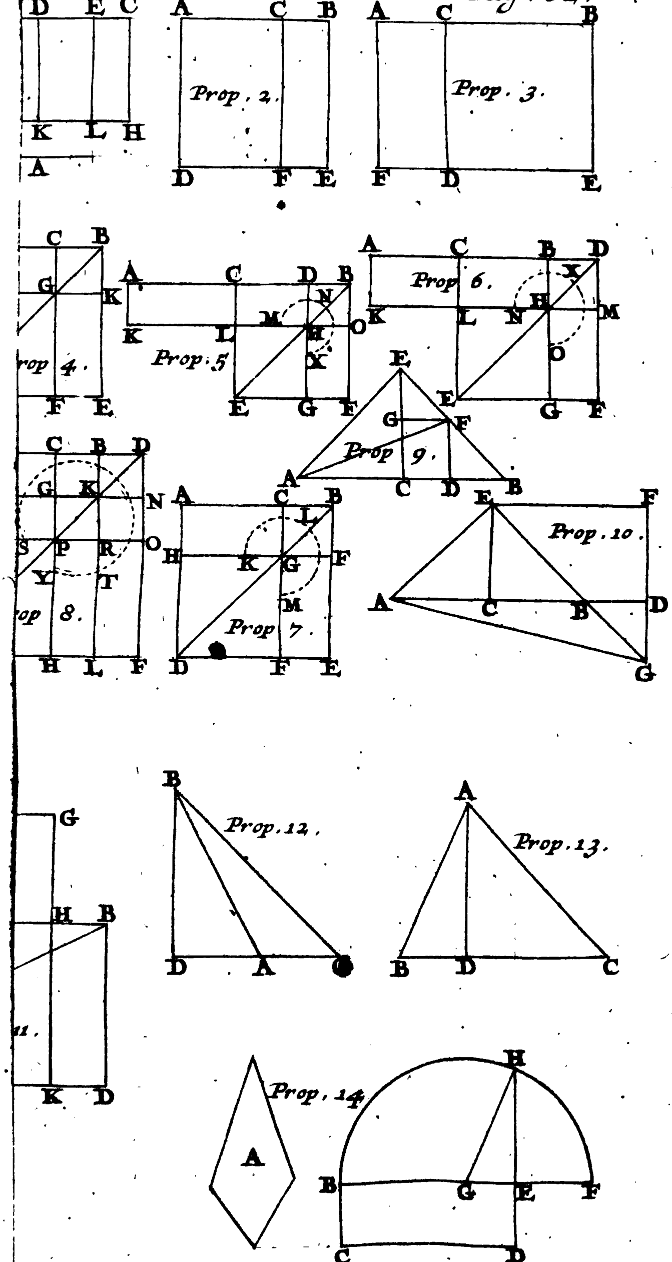
There-

* 45. 1.
† 10. 1.
‡ 5 of this
* 47. 1.

Therefore the Parallelogram BD is equal to the Square of EH; but the Parallelogram BD is equal to the Right-lin'd Figure A. Wherefore the Right-lin'd Figure A is equal to the Square of EH. And so there is a Square made equal to the given Right-lin'd Figure A, *viz.* the Square of EH; *which was to be done.*

The END *of the* SECOND BOOK.

EUCLID's


Prop. 2.
Prop. 3.
Prop 4.
Prop. 5
Prop 6.
Prop 9.
Prop 8.
Prop 7.
Prop. 10.
11.
Prop. 12.
Prop. 13.
Prop. 14

EUCLID's ELEMENTS.

BOOK III.

DEFINITIONS.

I. *EQUAL Circles are such whose Diameters are equal; or from whose Centers the Right Lines that are drawn are equal.*

II. *A Right Line is said to touch a Circle, when touching the same, and being produc'd, does not cut it.*

III. *Circles are said to touch each other, which Touching do not cut one another.*

IV. *Right Lines in a Circle are said to be equally distant from the Center, when Perpendiculars drawn from the Center to them be equal.*

V. *And that Line is said to be farther from the Center, on which the greater Perpendicular falls.*

VI. *A Segment of a Circle is a Figure contained under a Right Line, and a Part of the Circumference of a Circle.*

VII. *An Angle of a Segment is that which is contained by a Right Line, and the Circumference of a Circle.*

VIII. *An Angle is said to be in a Segment, when some Point is taken in the Circumference thereof, and from it Right Lines are drawn to the Ends of that Right Line, which is the Base of the Segment; then the Angle contained under the Lines drawn, is said to be an Angle in a Segment.*

IX. *But when the Right Lines containing the Angle do receive any Circumference of the Circle, then the Angle is said to stand upon that Circumference.*

X. *A Sector of a Circle, is that Figure comprehended between the Right Lines drawn from the Center, and the Circumference contained between them.*

XI. *Similar Segments of Circles are those which include equal Angles, or whereof the Angles in them are equal.*

PROPOSITION I.

PROBLEM.

To find the Center of a Circle given.

LET ABC be the Circle given. It is required to find the Center thereof.

Let the Right Line AB be any how drawn in it, which * bisect in the Point D; and let DC be † drawn from the Point D at Right Angles to AB, which let be produced to E.

* 10. 1.
† 11. 1.

Then if EC be * bisected in F, I say, the Point F is the Center of the Circle ABC.

For if it be not, let G be the Center, and let GA, GD, GB, be drawn. Now because DA is equal to DB, and DG is common, the two Sides AD, DG, are equal to the two Sides GD, DB, each to each; and the Base GA is ‡ equal to the Base GB; for they are drawn from the Center G. Therefore the Angle ADG is ‖ equal to the Angle GDB. But when a Right Line standing upon a Right Line, makes the adjacent

‡ Def. 15. 1.
‖ 8. 1.

adjacent Angles equal to one another, each of the equal Angles will * be a Right Angle. Wherefore the Angle GDB is a Right Angle. But FDB is also a Right Angle. Therefore the Angle FDB is equal to the Angle GDB, a greater to a less, which is absurd. Wherefore G is not the Center of the Circle ABC. After the same Manner we prove, that no other Point, unless F, is the Center. Therefore F is the Center of the Circle ABC; *which was to be found.* *Def. 10. 1.

Coroll. If in a Circle, any Right Line cuts another Right Line into two equal Parts, and at Right Angles; the Center of the Circle will be in that cutting Line.

PROPOSITION II.

THEOREM.

If any two Points be assum'd in the Circumference of a Circle, the Right Line joining those two Points shall fall within the Circle.

LET ABC be a Circle; in the Circumference of which let any two Points A, B, be assumed. I say, a Right Line drawn from the Point A to the Point B, falls within the Circle.

For let any Point E be taken in the Right Line AB, and let DA, DE, DB, be joined.

Then because DA is equal to DB, the Angle DAB will be * equal to the Angle DBA; and since the Side AE of the Triangle DAE is produced, the Angle DEB will be † greater than the Angle DAE, but the Angle DAE is equal to the Angle DBE; therefore the Angle DEB is greater than the Angle DBE. But the greater Side subtends the greater Angle. Wherefore DB is greater than DE. But DB only comes to the Circumference of the Circle; therefore DE does not reach so far. And so the Point E falls within the Circle. Therefore, *if two Points are assumed in the Circumference of a Circle, the Right Line joining those two Points shall fall within the Circle;* which was to be demonstrated. * 5. 1. † 16. 1.

Coroll. Hence if a Right Line touches a Circle, it will touch it in one Point only.

PROPOSITION. III.

THEOREM.

If in a Circle a Right Line drawn through the Center, cuts any other Right Line not drawn through the Center, into equal Parts, it shall cut it at Right Angles; and if it cuts it at Right Angles, it shall cut it into two equal Parts.

LET ABC be a Circle, wherein the Right Line CD, drawn thro' the Center, bisects the Right Line AB not drawn thro' the Center. I say, it cuts it at Right Angles.

* 1 of this. For * find E the Center of the Circle, and let EA, EB, be joined.

Then because AF is equal to FB, and FE is common, the two Sides AF, FE are equal to the two Sides BF, FE, each to each, but the Base EA is equal to the Base EB. Wherefore the Angle AFE

† 8. 1. shall be † equal to the Angle BFE. But when a Right Line standing upon a Right Line makes the adjacent Angles equal to one another, each of the equal Angles is ‡ a Right Angle. Wherefore AFE, or BFE, is a Right Angle. And therefore the Right Line CD drawn thro' the Center, bisecting the Right Line AB not drawn thro' the Center, cuts it at Right Angles. Now if CD cuts AB at Right Angles, I say, it will bisect it, that is, AF will be equal to FB. For the same Construction remaining, because EA, being drawn from the Center, is equal to EB, the Angle EAF

‖ 5. 1. shall be ‖ equal to the Angle EBF. But the Right Angle AFE is equal to the Right Angle BFE; therefore the two Triangles EAF, EBF, have two Angles of the one equal to two Angles of the other, and the Side EF is common to both. Wherefore the other Sides of the one shall be ⸸ equal to the other Sides of the

⸸ 26. 1. other: And so AF will be equal to FB. Therefore *if in a Circle a Right Line drawn thro' the Center, cuts any other Right Line not drawn thro' the Center, into two equal Parts, it shall cut it at Right Angles; and if it*

it cuts it at Right Angles, it shall cut it into two equal Parts; which was to be demonstrated.

PROPOSITION IV.

THEOREM.

If in a Circle two Right Lines, not being drawn thro' the Center, cut each other, they will not cut each other into two equal Parts.

LET ABCD be a Circle, wherein two Right Lines AC, BD, not drawn thro' the Center, cut each other in the Point E. I say, they do not bisect each other.

For, if possible, let them bisect each other, so that AE be equal to EC, and BE to ED. Let the Center F of the Circle ABCD be * found, and join EF. * 1 *of this.*

Then because the Right Line FE drawn thro' the Center, bisects the Right Line AC not drawn thro' the Center, it will † cut AC at Right Angles. And † 3 *of this.* so FEA is a Right Angle. Again, because the Right Line FE bisects the Right Line BD not drawn thro' the Center, it will † cut BD at Right Angles. Therefore FEB is a Right Angle. But FEA has been shewn to be also a Right Angle. Wherefore the Angle FEA will be equal to the Angle FEB, a less to a greater; which is absurd. Therefore AC, BD, do not mutually bisect each other. And so *if in a Circle two Right Lines, not being drawn thro' the Center, cut each other, they will not cut each other into two equal Parts*; which was to be demonstrated.

PROPOSITION. V.

THEOREM.

If two Circles cut one another, they shall not have the same Center.

LET the two Circles ABC, CDG, cut each other in the Points B, C. I say, they have not the same Center.

For if they have, let it be E, and join EC, and draw EFG at pleasure.

Now because E is the Center of the Circle CDG, CE will be equal to EF. Again, because E is the Center of the Circle CDG, CE is equal to EG. But CE has been ſhewn to be equal to EF. Therefore EF ſhall be equal to EG, a leſs to a greater, which cannot be. Therefore the Point E is not the Center of both the Circles ABC, CDG. Wherefore, *if two Circles cut one another, they ſhall not have the ſame Center*; which was to be demonſtrated.

PROPOSITION VI.

THEOREM.

If two Circles touch one another inwardly, they will not have one and the ſame Center.

LET two Circles ABC, CDE, touch one another inwardly in the Point C. I ſay, they will not have one and the ſame Center.

For if they have, let it be F, and join FC, and draw FB any how.

Then becauſe F is the Center of the Circle ABC, CF is equal to FB. And becauſe F is only the Center of the Circle CDE, CF ſhall be equal to EF. But CF has been ſhewn to be equal to FB. Therefore FE is equal to FB, a leſs to a greater; which cannot be. Therefore the Point F is not the Center of both the Circles ABC, CDE. Wherefore, *if two Circles touch one another inwardly, they will not have one and the ſame Center*; which was to be demonſtrated.

PRO-

PROPOSITION VII.

THEOREM.

If in the Diameter of a Circle some Point be taken, which is not the Center of the Circle, and from that Point certain Right Lines fall on the Circumference of the Circle, the greatest of these Lines shall be that wherein the Center of the Circle is; the least, the Remainder of the same Line. And of all the other Lines, the nearest to that which was drawn thro' the Center, is always greater than that more remote, and only two equal Lines fall from the abovesaid Point upon the Circumference, on each Side of the least or greatest Lines.

LET ABCD be a Circle, whose Diameter is AD, in which assume some Point F, which is not the Center of the Circle. Let the Center of the Circle be E; and from the Point F let certain Right Lines FB, FC, FG, fall on the Circumference. I say, FA is the greatest of these Lines, and FD the least; and of the others FB is greater than FC, and FC greater than FG.

For let BE, CE, GE, be joined.

Then because two Sides of every Triangle are * greater than the third; BE, EF, are greater than BF. But AE is equal to BE. Therefore BE and EF are equal to AF. And so AF is greater than FB. * 20. 1.

Again, because BE is equal to CE, and FE is common, the two Sides BE and FE, are equal to the two Sides CE, EF. But the Angle BEF is greater than the Angle CEF. Wherefore the Base BF is greater than the Base FC †. For the same Reason, CF is greater than FG. † 24. 1.

Again, because GF and FE are † greater than GE, and GE is equal to ED; GF and FE shall be greater than ED; and if FE, which is common, be taken away, then the Remainder GF is greater than the Remainder FD. Wherefore FA is the greatest of the Right Lines, and FE the least: Also BF is greater than FC, and FC greater than FG.

 I say,

I say, moreover, that there are only two equal Right Lines that can fall from the Point F on A B C D, the Circumference of the Circle, on each Side the shortest Line F D. For at the given Point E, with the Right Line E F, make ‡ the Angle F E H equal to the Angle G E F, and join F H. Now because G E is equal to E H, and E F is common, the two Sides G E and E F, are equal to the two Sides H E and E F. But the Angle G E F, is equal to the Angle H E F. Therefore the Base F G shall be ‡ equal to the Base F H. I say, no other Right Line falling from the Point F, on the Circle, can be equal to F G. For if there can, let this be F K. Now since F K is equal to F G, as also F H, F K will be equal to F H, *viz*, a Line drawn nigher to that passing thro' the Center, equal to one more remote, which cannot be. *If therefore, in the Diameter of a Circle, some Point be taken, which is not the Center of the Circle, and from that Point certain Right Lines fall on the Circumference of the Circle, the greatest of these Lines shall be that wherein the Center of the Circle is; the least, the Remainder of the same Line. And of all the other Lines, the nearest to that which was drawn thro' the Center, is always greater than that more remote; and only two equal Lines fall from the abovesaid Point upon the Circumference, on each Side of the least or greatest Lines*; which was to be demonstrated.

‡ 23. 1

PRO-

PROPOSITION VIII.

THEOREM.

If some Point be assum'd without a Circle, and from it certain Right Lines be drawn to the Circle, one of which passes thro' the Center, but the other any how; the greatest of these Lines, is that passing thro' the Center, and falling upon the Concave Part of the Circumference of the Circle; and of the others, that which is nearest to the Line passing thro' the Center is greater than that more remote. But the least of the Lines that fall upon the Convex Circumference of the Circle, is that which lies between the Point and the Diameter; and of the others, that which is nigher to the least, is less than that which is further distant; and from that Point there can be drawn only two equal Lines, which shall fall on the Circumference on each Side the least Line.

LET ABC be a Circle, out of which take any Point D. From this Point let there be drawn certain Right Lines D A, D E, D F, D C, to the Circle, whereof D A passes thro' the Center. I say D A, which passes thro' the Center, is the greatest of the Lines falling upon A E F C, the Concave Circumference of the Circle, and the least is D G, *viz.* the Line drawn from D to the Diameter G A: Likewise D E is greater than D F, and D F greater than D C. But of these Lines that fall upon H L G K the Convex Circumference of the Circle, that which is nearest the least D G, is always less than that more remote; that is, D K is less than D L, and D L less than D H.

For find * M the Center of the Circle A B C, and let M E, M F, M C, M H, M L, be join'd. * 1. of this.

Now because A M is equal to M E; if M D, which is common, be added, A D will be equal to E M and M D. But E M and M D are † greater than E D † therefore A D is also greater than E D. Again, because M E is equal to M F, and M D is common, then E M, M D, shall be equal to M F, M D; and the Angle E M D is greater than the An- † 20. 1.

† 24. 1. gle FMD. Therefore the Base ED will be † greater than the Base FD. We prove, in the same Manner that FD is greater than CD. Wherefore DA is the greatest of the Right Lines falling from the Point D; DE is greater than DF, and DF is greater than DC.

* 20. 1. Moreover, because MK and KD are * greater than MD, and MG is equal to MK; then the Remain- † axiom. 4. der KD will † be greater than the Remainder GD. And so GD is less than KD, and consequently is the least. And because two Right Lines MK, KD, are drawn from M and D to the Point K, with- ‡ 21. 1. in the Triangle MLD, MK, and KD, are ‡ less than ML and LD; but MK is equal to KL. Wherefore the Remainder DK is less than the Remainder DL. In like Manner we demonstrate that DL is less than DH. Therefore DG is the least. And DK is less than DL, and DL than DH.

I say, likewise, that from the Point D only two equal Right Lines can fall upon the Circle on each * 23. 1. Side the least Line. For make * the Angle DMB at the Point M, with the Right Line MD, equal to the Angle KMD, and join DB. Then because MK is equal to MB, and MD is common, the two Sides KM, MD, are equal to the two Sides BM, MD, each to each; but the Angle KMD is equal to the Angle BMD. Therefore the Base † 4. 1. DK is † equal to the Base DB. I say no other Line can be drawn from the Point D to the Circle equal to DK; for if there can, let DN. Now since DK is equal to DN, as also to DB, therefore DB shall be equal to DN, *viz.* the Line drawn nearest to the least equal to that more remote, which has been shewn to be impossible. Therefore, *if some Point be assum'd without a Circle, and from it certain Right Lines be drawn to the Circle, one of which passes through the Center, but the others any how; the greatest of these Lines, is that passing thro' the Center, and falling upon the Concave Part of the Circumference of the Circle; and of the others, that which is nearest to the Line passing thro' the Center, is greater than that more remote. But the least of the Lines that fall upon the Convex Circumference of the Circle, is that which lies between the Point and the Diameter; and of the others,*

that

that which is nigher to the least, is less than that which is further distant; and from that Point there can be drawn only two equal Lines which shall fall on the Circumference on each Side the least Line; which was to be demonstrated.

PROPOSITION. IX.

THEOREM.

If a Point be assum'd in a Circle, and from it more than two equal Right Lines be drawn to the Circumference; then that Point is the Center of the Circle.

LET the Point D be assum'd within the Circle ABC; and from the Point D, let there fall more than two Right Lines to the Circumference, *viz.* the Right Lines DA, DB, DC. I say the assum'd Point D is the Center of the Circle ABC.

For if it be not, let E be the Center, if possible, and join DE, which produce to G and F.

Then FG is a Diameter of the Circle ABC; and so because the Point D, not being the Center of the Circle, is assum'd in the Diameter FG, DG, will * be the greatest Line drawn from D to the Circumference, and DC greater than DB, and DB than DA; but they are also equal, which is absurd. Therefore E is not the Center of the Circle ABC. And in this Manner we prove that no other Point except D is the Center; therefore D is the Center of the Circle ABC; *which was to be demonstrated.* * 7 *of this.*

Otherwise:

Let ABC be the Circle, within which take the Point D, from which let more than two equal Right Lines fall on the Circumference of the Circle, *viz.* the three equal ones DA, DB, DC: I say, the Point D is the Center of the Circle ABC.

For join AB, BC, which bisect * in the Points E and Z; as also join ED, DZ; which produce to the Points H, K, O, L; then because AE is equal to EB, and ED is common, the two Sides AE, ED, shall

shall be equal to the two Sides BE, ED. And the Base DA is equal to the Base DB: Therefore the
* 8. 1. Angle AED will be * equal to the Angle BED: And so [by *Def.* 10. 1.] each of the Angles AED, BED, is a Right Angle: Therefore HK bisecting AB, cuts it at Right Angles. And because, a Right Line in a Circle, bisecting another Right Line, cuts it at Right Angles, and the Center of the Circle is in the cutting Line, [by *Cor.* 1. 3.] the Center of the Circle ABC will be in HK. For the same Reason, the Center of the Circle will be in OL. And the Right Lines HK, OL, have no other Point common but D: Therefore D is the Center of the Circle ABC; *which was to be demonstrated.*

PROPOSITION X.

THEOREM.

A Circle cannot cut another Circle in more than two Points.

FOR if it can, let the Circle ABC cut the Circle DEF in more than two Points, *viz.* in B, G, F, and let K be the Center of the Circle ABC, and join KB, KG, KF.

Now because the Point K is assum'd within the Circle DEF, from which more than two equal Right Lines KB, KG, KF, fall on the Circumference, the Point K shall be † the Center of the Circle DEF. But K is ‡ the Center of the Circle ABC. Therefore K will be the Center of two Circles cutting each other, which is absurd. Wherefore a Circle cannot cut a Circle in more than two Points; *which was to be demonstrated.*

† 9. *of this.*
‡ *By Hyp.*

PRO-

PROPOSITION XI.

THEOREM.

If two Circles touch each other on the Inside, and the Centers be found, the Line joining their Centers, will fall on the [Point of] Contact of those Circles.

LET two Circles ABC, ADE, touch one another inwardly in A, and let F be the Center of the Circle ABC, and G that of ADE. I ſay, a Right Line joining the Centers G and F, being produc'd, will fall in the Point A.

If this be denied, let the Right Line, joining FG, cut the Circle in D and H.

Now becauſe A G, G F, are greater than A F, * that is, than FH; take way F G, which is common, and the Remainder A G is greater than the Remainder G H. But A G is equal to G D; therefore G D is greater than G H, the leſs than the greater, which is abſurd. Wherefore a Line drawn thro' the Points F, G, will not fall out of the Point of Contact A, and ſo neceſſarily muſt fall in it; *which was to be demonſtrated.* * 12. 1.

PROPOSITION XII.

THEOREM.

If two Circles touch one another on the Outſide, a Right Line joining their Centers will paſs thro' the [Point of] Contact.

LET two Circles ABC, ADE, touch one another outwardly in the Point A; and let F be the Center of the Circle ABC, and G that of ADE. I ſay, a Right Line drawn thro' the Centers F, G will paſs thro' the Point of Contact A.

For if it does not, let, if poſſible, FCDG fall without it, and join FA, AG.

Now ſince F is the Center of the Circle ABC, AF will be equal FC. And becauſe G is the Center of the

the Circle A D E, A G will be equal to G D: But A F has been shewn to be equal to F C; therefore F A, A G, are equal to F C, D G. And so the whole
* 20. 1. F G is greater than F A, A G; and also less, * which is absurd. Therefore a Right Line drawn from the Point F to G, will pass thro' the Point of Contact A; *which was to be demonstrated.*

PROPOSITION XIII.

THEOREM.

One Circle cannot touch another in more Points than one, whether it be inwardly, or outwardly.

FOR, in the first Place, if this be denied, let the Circle A B D C, if possible, touch the Circle E B F D inwardly, in more Points than one, *viz.* in B, D.

And let G be the Center of the Circle A B D C, and H that of E B F D.

Then a Right Line drawn from the Point G to H,
† 11 *of this.* will † fall in the Points B and D. Let this Line be B G H D. And because G is the Center of the Circle A B D C, the Line B G will be equal to G D. Therefore B G is greater than H D, and B H much greater than H D. Again, since H is the Center of the Circle E B F D, the Line B H is equal to H D. But it has been prov'd to be much greater than it, which is absurd. Therefore one Circle cannot touch another Circle inwardly in more Points than one.

Secondly, let the Circle A C K, if possible, touch the Circle A B D C outwardly in more Points than one, *viz.* in A and C, and let A, C, be join'd.

Now because two Points A, C, are assum'd in the Circumference of each of the Circles A B D C, A C K
‡ 2 *of this.* a Right Line joining these two Points, will fall ‡ within either of the Circles. But it falls within the Circle A B D C, and without the Circle A C K, which is absurd. Therefore one Circle cannot touch another Circle in more Points than one outwardly. But it has been prov'd, that one Circle cannot touch another Circle inwardly, [in more Points than one.]

Where-

Wherefore *one Circle cannot touch another in more Points than one, whether it be inwardly or outwardly*; which was to be demonſtrated.

PROPOSITION XIV.

THEOREM.

Equal Right Lines in a Circle are equally diſtant from the Center; and Right Lines, which are equally diſtant from the Center, are equal between themſelves.

LET ABDC be a Circle, wherein are the equal Right Lines AB, CD. I ſay theſe Lines are equally diſtant from the Center of the Circle.

For let E be the Center of the Circle ABDC; from which let there be drawn EF, EG, perpendicular to AB, CD, and let AE, EC, be joined.

Then becauſe a Right Line EF, drawn thro' the Center, cuts the Right Line AB, not drawn thro' the Center at Right Angles, it will * biſect the ſame, * 4 of this. Wherefore AF is equal to FB, and ſo AB is double to AF. For the ſame Reaſon CD is double to CG, but AB is equal to CD. Therefore AF is equal to CG; and becauſe AE is equal to EC, the Square of AE ſhall be equal to the Square of EC. But the Squares of AF and FE are † equal to the Square of † 47. 1. AE. For the Angle at F is a Right Angle; and the Squares of EG, and GC, are equal to the Square of EC, ſince the Angle at G is a Right one. Therefore the Squares of AF and FE, are equal to the Squares of CG and GE. But the Square of AF is equal to the Square of CG; for AF is equal to CG. Therefore the Square of FE is equal to the Square of EG; and ſo FE equal to EG. Alſo Lines in a Circle are ‡ ſaid to be equally diſtant from the Center, ‡ Def. 4 of this. when Perpendiculars drawn to them from the Center are equal. Therefore AB, CD, are equally diſtant from the Center.

But if AB, CD, are equally diſtant from the Center, that is, if FE be equal to EG. I ſay AB, is equal to CD.

For the ſame Conſtruction being ſuppos'd, we demonſtrate as above, that AB is double to AF, and

CD

CD to CG; and becauſe AE is equal to EC, the Square of AE will be equal to the Square of EC.
† 47. 1. But the Squares of EF and FA, are † equal to the Square of AE, and the Squares of EG, and GC, equal † to the Squares of EC. Therefore the Squares of EF, and FA, are equal to the Squares of EG and GC. But the Square of EG is equal to the Square of EF; for EG is equal to EF. Therefore the Square of AF is equal to the Square of CG; and ſo AF is equal to CG. But AB is double to AF, and CD to CG. Therefore *equal Right Lines in a Circle are equally diſtant from the Center; and Right Lines, which are equally diſtant from the Center, are equal between themſelves*; which was to be demonſtrated.

PROPOSITION XV.

THEOREM.

A Diameter is the greateſt Line in a Circle; and of all the other Lines therein, that which is neareſt to the Center is greater than that more remote.

LET ABCD be a Circle, whoſe Diameter is AD, and center E; and let BC be nearer to the Diameter than FG. I ſay, AD is the greateſt, and BC is greater than FG.

For let the Perpendiculars EH, EK, be drawn from the Center E to BC, FG. Now becauſe BC is nearer to the Center than FG, EK will be greater than EH. Let EL be equal to EH; draw LM thro' L at Right Angles to EK, which produce to N, and let EM, EN, EF, EG, be join'd.

Then becauſe EH is equal to EL, the Line BC
* 14 of this. will be equal to MN*. Again, ſince AE is equal to EM, and DE to EN, AD will be equal to ME
† 20. 1. and EN. But ME and EN are † greater than MN: And ſo AD is greater than MN; and NM is equal to BC. Therefore AD is greater than BC. And ſince the two Sides EM, EN, are equal to the two Sides FE, EG, and the Angle MEN greater than
‡ 24. 1. the Angle FEG; the Baſe MN ſhall be ‡ greater than the Baſe FG. But MN is equal to BC. Therefore

fore BC is greater than FG. And so the Diameter AD is the greatest, and BC is greater than FG. Wherefore *the Diameter is the greatest Line in a Circle; and of all the other Lines therein, that which is nearest to the Center is greater than that more remote*; which was to be demonstrated.

PROPOSITION XVI.

THEOREM.

A Line drawn from the extreme [Point] of the Diameter of a Circle at Right Angles to that Diameter, shall fall without the Circle; and between the said Right Line, and the Circumference, no other Line can be drawn; and the Angle of a Semicircle is greater than any Right-lin'd acute Angle; and the remaining Angle [without the Circumference] is less than any Right-lin'd Angle.

LET ABC be a Circle, whose Center is D, and Diameter AB. I say, a Right Line drawn from the Point A at Right Angles to AB, falls without the Circle.

For if it does not, let it fall, if possible, within the Circle, as AC, and join DC.

Now because DA is equal to DC, the Angle DAC shall be * equal to the Angle ACD. But DAC is a Right Angle; therefore ACD is a Right Angle: And accordingly the Angles DAC, ACD, are equal to two Right Angles; which is absurd †. Therefore a Right Line drawn from the Point A at Right Angles to BA, will not fall within the Circle; and so likewise we prove, that it neither falls in the Circumference. Therefore it will necessarily fall without the same; which now let be AE.

* 5. 1.

† 17. 1.

Again, between the Right Line AE, and the Circumference CHA, no other Right Line can be drawn.

For if there can, let it be FA, and let ADG be ‡ drawn at Right Angles from the Center D to FA.

‡ 12. 1.

Now because AGD is a Right Angle, and DAG is less than a Right Angle, DA will be greater than DG *. But DA is equal to DH. Therefore DH is greater

* 19. 1.

greater than DG, the less than the greater; which is absurd. Wherefore no Right Line can be drawn between AE, and the Circumference AHC. I say moreover, that the Angle of the Semicircle, contain'd under the Right Line BA, and the Circumference CHA, is greater than any Right-lin'd acute Angle; and the remaining Angle contain'd under the Circumference CHA, and the Right Line AE, is less than any Right-lin'd Angle.

For if any Right-lin'd acute Angle be greater than the Angle contain'd under the Right Line BA, and the Circumference CHA; or if any Right-lin'd Angle be less than that contain'd under the Circumference CHA, and the Right Line AE, then a Right Line may be drawn between the Circumference CHA and the Right Line AE, making an Angle greater than that contain'd under the Right Line BA, and the Circumference CHA, *viz.* which is contain'd under Right Lines, and less than that contain'd under the Circumference CHA, and the Right Line AE. But such a Right Line cannot be drawn from what has been prov'd. Therefore no Right-lin'd acute Angle, is greater than the Angle contain'd under the Right Line BA, and the Circumference CHA; nor less than the Angle contain'd under the Circumference CHA, and the Right Line AE.

Coroll. From hence it is manifest, that a Right Line drawn at Right Angles on the End of the Diameter of a Circle, touches the Circle, and that in one Point only, because, if it should meet it in two Points, it would fall within the same; *as has been demonstrated.*

PROPOSITION XVII.

PROBLEM.

To draw a Right Line from a given Point, that shall touch a given Circle.

LET A be the Point given, and BCD the Circle. It is requir'd to draw a Right Line from the Point A, that shall touch the given Circle BCD.

Let E be the Center of the Circle, and join AE; then about the Center E, with the Distance EA, describe the Circle AFG; draw DF * at Right Angles to EA, and join EBF, and AB. I say the Right Line AB is drawn from the Point A, touching the Circle BCD. * 11. 1.

For since E is the Center of the Circles BCD, AFG, the Line EA will be equal to EF, and ED to EB. Therefore the two Sides AE, EB, are equal to the two Sides FE, ED, each to each; and they contain the common Angle E. Wherefore the Base DF is † equal to the Base AB, and the Triangle DEF equal to the Triangle EBA, and the remaining Angles of the one equal to the remaining Angles of the other. And so the Angle EBA is equal to the Angle EDF. But EDF is a Right Angle. Wherefore EBA is also a Right Angle, and EB is a Line drawn from the Center; but a Right Line drawn from the Extremity of the Diameter of a Circle at Right Angles ‡ to it, touches the Circle. Wherefore AB touches the Circle; *which was to be done.*

† 32. 1.

‡ *Cor.* 16 *of this.*

PROPOSITION XVIII.

THEOREM.

If any Right Line touches a Circle, and from the Center to the Point of Contact a Right Line be drawn; that Line will be perpendicular to the Tangent.

LET any Right Line DE touch a Circle ABC in the Point C, and let there be drawn the Right Line FC from the Center C. I say FC is perpendicular to DE.

For if it be not, let FG be drawn * from the Point F, perpendicular to DE. * 12. 1.

Now because the Angle FGC is a Right Angle, the Angle GCF will be † an acute Angle; and accordingly the Angle FGC is greater than the Angle FCG; but the greater Side subtends ‡ the greater Angle. Therefore FC is greater than FG. But FC equal to FB. Wherefore FB is greater than FG, a less than the greater; which is absurd. Therefore FG is not perpendicular to DE. And in the same

† 32. 1.

‡ 19. 1.

Manner we prove, that no other Right Line but FC is perpendicular to DE. Wherefore FC is perpendicular to DE. Therefore, *if any Right Line touches a Circle, and from the Center to the Point of Contact a Right Line be drawn; that Line will be perpendicular to the Tangent*; which was to be demonſtrated.

PROPOSITION XIX.

THEOREM.

If any Right Line touches a Circle, and from the Point of Contact a Right Line be drawn at Right Angles to the Tangent, the Center of the Circle ſhall be in the ſaid Line.

LET any Right Line DE touch the Circle ABC in C, and let CA be drawn from the Point C at Right Angles to DE. I ſay, the Circle's Center is in AC.

For if it be not, let F be the Center, if poſſible, and join CF.

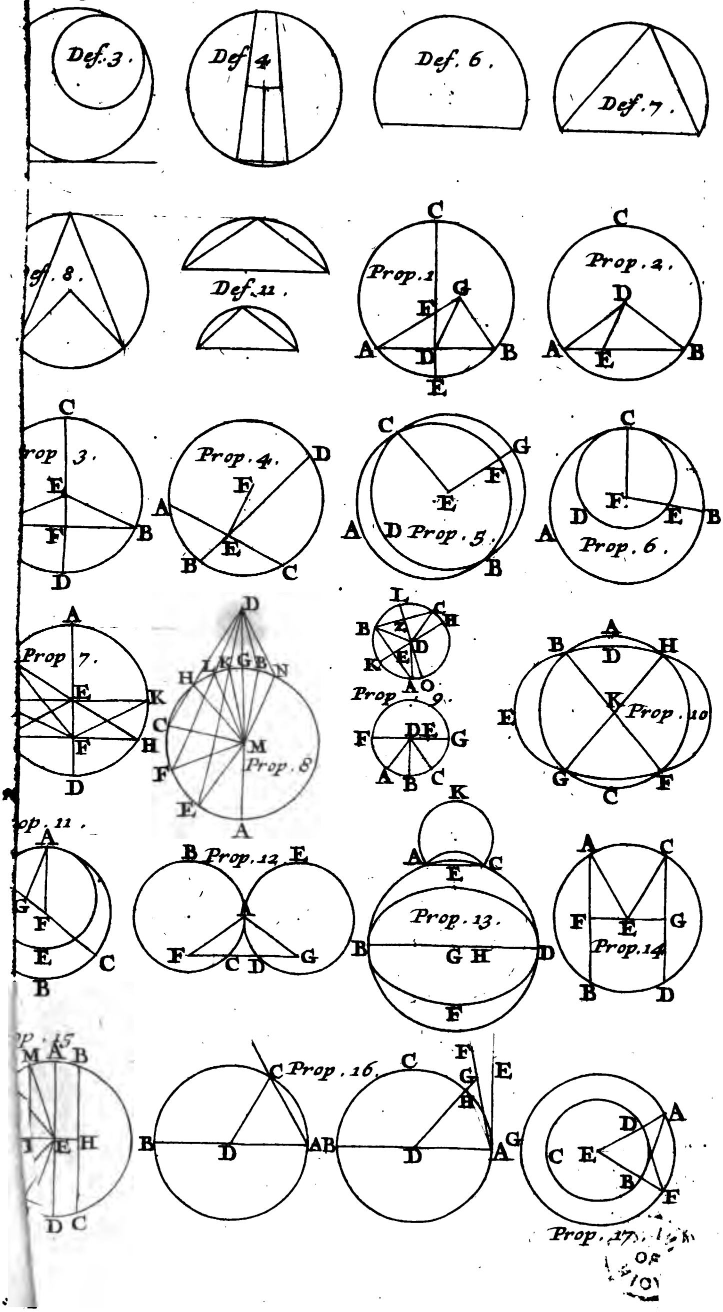
Then becauſe the Right Line DE touches the Circle ABC, and FC is drawn from the Center to the Point of Contact; FC will be perpendicular to DE*. And ſo the Angle FCE is a Right one. But ACE is alſo a Right Angle †: Therefore the Angle FCE is equal to the Angle ACE, a leſs to a greater; which is abſurd. Therefore F is not the Center of the Circle ABC. After this Manner we prove, that the Center of the Circle can be in no other Line, unleſs in AC. Wherefore, *if any Right Line touches a Circle, and from the Point of Contact a Right Line be drawn at Right Angles to the Tangent, the Center of the Circle ſhall be in the ſaid Line*; which was to be demonſtrated.

* 18 *of this.*

† *From the Hyp.*

PRO-

Book 3. Plate 1.



PROPOSITION XX.

THEOREM.

The Angle at the Center of a Circle is double to the Angle at the Circumference, when the same Arc is the Base of the Angles.

LET ABC be a Circle, at the Center, whereof is the Angle BEC, and at the Circumference, the Angle BAC, both of which stand upon the same Arc BC. I say, the Angle BEC is double to the Angle BAC.

For join AE and produce it to F.

Then because EA is equal to EB, the Angle EAB shall be equal to the Angle EBA*. Therefore the Angles EAB, EBA, are double to the Angle EAB; but the Angle BEF is † equal to the Angles EAB, EBA; therefore the Angle BEF is double to the Angle EAB. For the same Reason, the Angle EFC is double to EAC. Therefore the whole Angle BEC is double to the whole Angle BAC. Again, let there be another Angle BDC, and join DE, which produce to G. We demonstrate in the same Manner, that the Angle GEC is double to the Angle GDC; whereof the Part GEB is double to the Part GDB. And therefore BEC is double to BDC. Consequently, *an Angle at the Center of a Circle is double to the Angle at the Circumference, when the same Arc is the Base of the Angles*; which was to be demonstrated.

* 5. 1.
† 32. 1.

PROPOSITION XXI.

THEOREM.

Angles that are in the same Segment in a Circle, are equal to each other.

LET ABCDE be a Circle, and let BAD, BED, be Angles in the same Segment BAED. I say, those Angles are equal.

For let F be the Center of the Circle ABCDE, and join BF, FD.

Now because the Angle BFD is at the Center, and the Angle BAD at the Circumference, and they stand upon the same Arc BCD; the Angle BFD
* 20 of this. will be * double to the Angle BAD. For the same Reason, the Angle BFD is also double to the Angle BED. Therefore the Angle BAD will be equal to the Angle BED.

If the Angles BAD, BED, are in a Segment less than a Semicircle, let AE be drawn; and then all the
† 32. 1. Angles of the Triangle ABG are † equal to all the Angles of the Triangle DEG. But the Angles ABE, ADE, are equal, from what has been before prov'd,
‡ 15. 1. and the Angles AGB, DGE, are also equal ‡, for they are vertical Angles. Wherefore the remaining Angle BAG is equal to the remaining Angle GED. Therefore, *Angles that are in the same Segment in a Circle, are equal to each other*; which was to be demonstrated.

PROPOSITION XXII.

THEOREM.

The opposite Angles of any Quadrilateral Figure described in a Circle, are equal to two Right Angles.

LET ABDC be a Circle, wherein is described the Quadrilateral Figure ABCD. I say, two opposite Angles thereof are equal to two Right Angles.

For join AD, BC.

Then because the three Angles of any Triangle are
* 32. 1. * equal to two Right Angles, the three Angles of the Triangle ABC, *viz.* the Angles CAB, ABC, BCA, are equal to two Right Angles. But the Angle ABC
† 21 of this. is † equal to the Angle ADC; for they are both in the same Segment ABDC. And the Angle ACB is † equal to the Angle ADB, because they are in the same Segment ACDB: Therefore the whole Angle BDC is equal to the Angles ABC, ACB; and if the common Angle BAC be added, then the Angles BAC, ABC, ACB, are equal to the Angles BAC, BDC; but the Angles BAC, ABC, ACB, are

* equal to the two Right Angles. Therefore likewise, the Angles BAC, BDC shall be equal to two Right Angles. And after the same Way we prove, that the Angles ABD, ACD, are also equal to two Right Angles. Therefore *the opposite Angles of any Quadrilateral Figure described in a Circle, are equal to two Right Angles*; which was to be demonstrated. * 32. 1.

PROPOSITION XXIII.

THEOREM.

Two similar and unequal Segments of two Circles, cannot be set upon the same Right Line, and on the same Side thereof.

FOR if this be possible, let the two similar and unequal Segments ACB, ADB, of two Circles stand upon the Right Line AB on the same Side thereof. Draw ACD, and let CB, BD, be joined. Now because the Segment ACB is similar to the Segment ADB, and similar Segments of Circles are * such which receive equal Angles; the Angle ACB will be equal to the Angle ADB, the outward one to the inward one; which is † absurd. Therefore *similar and unequal Segments of two Circles, cannot be set upon the same Right Line, and on the same Side thereof*; which was to be demonstrated.

* *Def.* 11. *of this.*
† 16. 1.

PROPOSITION XXIV.

THEOREM.

Similar Segments of Circles being upon equal Right Lines, are equal to one another.

LET AEB, CFD be equal Segments of Circles, standing upon the equal Right Lines AB, CD. I say, the Segment AEB is equal to the Segment CFD.

For the Segment AEB being apply'd to the Segment CFD, so that the Point A co-incides with C, and the Line AB with CD; then the Point B will co-incide with the Point D, since AB and CD are equal. And since the Right Line AB co-incides with

CD, the Segment AEB will coincide with the Segment CFD. For if at the same Time that AB coincides with CD, the Segment AEB should not coincide with the Segment CFD, but be otherwise, as CGD; then a Circle would cut a Circle in more Points than two, *viz.* in the Points C, G, D; which
* 10 *of this.* is * impossible. Wherefore if the Right Line AB co-incides with CD, the Segment AEB will co-incide with and be equal to the Segment CFD. Therefore *similar Segments of Circles being upon equal Right Lines, are equal to one another*; which was to be demonstrated.

PROPOSITION XXV.

PROBLEM.

A Segment of a Circle being given, to describe the Circle whereof it is the Segment.

LET ABC be a Segment of a Circle given. It is requir'd to describe a Circle whereof ABC is a Segment.

* 10. 1. † 11. 1. Bisect * AC in D, and let DB be drawn † from the Point D at Right Angles to AC, and join AB. Now the Angle ABD is either greater, equal, or less than the Angle BAD. And first let it be greater, and
‡ 23. 1. make ‡ the Angle BAE at the given Point A, with the Right Line BA, equal to the Angle ABD; produce DB to E, and join EC.

Then because the Angle ABE is equal to the Angle BAE, the Right Line BE will be * equal to EA. And because AD is equal to DC, and DE common, the two Sides AD, DE, are each equal to the two Sides CD, DE; and the Angle ADE is equal to the Angle CDE; for each is a Right one. Therefore the Base AE is equal to the Base EC. But AE has been prov'd to be equal to EB. Wherefore BE is also equal to EC. And accordingly the three Right Lines AE, EB, EC, are equal to each other. Therefore a Circle describ'd about the Center E, with either of the Distances AE, EB, EC, shall pass thro' the other Points, and be that requir'd to be describ'd. But it is manifest that the Segment ABC is less than

a Semicircle, becauſe the Center thereof is without the ſame.

But if the Angle ABD be equal to the Angle BAD; then if AD be made equal to BD, or DC, the three Right Lines AD, DB, DC, are equal between themſelves, and D will be the Center of the Circle to be deſcrib'd; and the Segment ABC is a Semicircle.

But if the Angle ABD is leſs than the Angle BAD, let the Angle BAE be made, at the given Point A with the Right Line BA, within the Segment ABC, equal to the Angle ABD;

Then the Point E, in the Right Line DB, will be the Center, and ABC a Segment greater than a Semicircle. Therefore a Circle is deſcrib'd, whereof a Segment is given; *which was to be done.*

PROPOSITION XXVI.

THEOREM.

In equal Circles, equal Angles ſtand upon equal Circumferences, whether they be at their Centers, or at their Circumferences.

LET ABC, DEF, be equal Circles, and let BGC, EHF, be equal Angles at their Centers, and BAC, EDF, equal Angles at their Circumferences. I ſay the Circumference BKC is equal to the Circumference ELF.

For let BC, EF, be joined. Becauſe ABC, DEF, are equal Circles, the Lines drawn from their Centers will be equal. Therefore the two Sides BG, GC, are equal to the two Sides EH, HF; and the Angle G is equal to the Angle at H. Wherefore the Baſe BC is * equal to the Baſe EF. Again, becauſe the * 4. 1.
Angle at A is equal to that at D, the Segment BAC will be † ſimilar to the Segment EDF; and they are † *Def.* 11.
upon equal Right Lines BC, EF. But thoſe ſimilar Segments of Circles, that are upon equal Right Lines, are ‡ equal to each other. Therefore the Segment ‡ 24 *of this.*
BAC, will be † equal to the Segment EDF. But † *Def.* 11.
the whole Circle ABC, is equal to the whole Circle DEF. Therefore the remaining Circumference BKC,

shall be equal to the remaining Circumference ELF. Therefore *in equal Circles, equal Angles stand upon equal Circumferences, whether they be at their Centers, or at their Circumferences*; which was to be demonstrated.

PROPOSITION XXVII.

THEOREM.

Angles, that stand upon equal Circumferences in equal Circles, are equal to each other; whether they be at their Centers or Circumferences.

LET the Angles BGC, EHF, at the Centers of the equal Circles ABC, DEF, and the Angles BAC, EDF, at their Circumferences, stand upon the equal Circumferences BC, EF. I say the Angle BGC is equal to the Angle EHF, and the Angle BAC to the Angle EDF.

For if the Angle BGC be equal to the Angle EHF, it is manifest that the Angle BAC is also equal to the Angle EDF: But if not, let one of them be the
*23. 1. greater, as BGC, and make * the Angle BGK, at the Point G, with the Line BG, equal to the Angle
† 26 of this. EHF. But equal Angles stand † upon equal Circumferences, when they are at the Centers. Wherefore the Circumference BK, is equal to the Circumference EF. But the Circumference EF is equal to the Circumference BC. Therefore BK is equal to BC, a less to a greater; which is absurd. Wherefore the Angle BGC is not unequal to the Angle EHF; and so it must be equal to it. But the Angle at A is one half of the Angle BGC; and the Angle at D one half of the Angle EHF. Therefore the Angle at A is equal to the Angle at D. Wherefore *Angles, that stand upon equal Circumferences in equal Circles, are equal to each other, whether they be at their Centers or Circumferences*; which was to be demonstrated.

PRO-

PROPOSITION XXVIII.

THEOREM.

In equal Circles, equal Right Lines cut off equal Parts of the Circumferences; the greater equal to the greater, and the lesser equal to the lesser.

LET ABC, DEF, be equal Circles, in which are the equal Right Lines BC, EF, which cut off the greater Circumferences BAC, EDF, and the lesser Circumferences BGC, EHF. I say the greater Circumference BAC, is equal to the greater Circumference EDF, and the lesser Circumference BGC, to the lesser Circumference EHF.

For assume the Centers K and L of the Circles, and join BK, KC, EL, LF.

Because the Circles are equal, the Lines drawn from their Centers are * also equal. Therefore the two Sides BK, KC, are equal to the two Sides EL, LF; and the Base BC, is equal to the Base LF. Therefore the Angle BKC, is † equal to the Angle ELF. But equal Angles stand ‡ upon equal Circumferences, when they are at the Centers. Wherefore the Circumference BGC, is equal to the Circumference EHF, and the whole Circle ABC, equal to the whole Circle DEF; and so the remaining Circumference BAC, shall be equal to the remaining Circumference EDF. Therefore in equal Circles, equal Right Lines cut off equal Parts of the Circumferences; *which was to be demonstrated.*

* *Def.* 1.
† 8. 1.
‡ 26 *of this.*

PROPOSITION XXIX.

THEOREM.

In equal Circles, equal Right Lines subtend equal Circumferences.

LET there be two equal Circles ABC, DEF; and let the equal Circumferences BGC, EHF, be assumed in them, and BC, EF, joined. I say, the Right Line BC is equal to the Right Line EF.

For

* 1 of this. For find * the Centers of the Circles K, L, and join BK, KC, EL, LF.

Then because the Circumference BGC is equal to the Circumference EHF, the Angle BKC shall be
† 27 of this. † equal to the Angle ELF. And because the Circles ABC, DEF, are equal, the Lines drawn from their
‡ Def. 1. Centers shall be ‡ equal. Therefore the two Sides BK, KC, are equal to the two Sides EL, LF; and they contain equal Angles: Wherefore the Base
‖ 4. 1. BC is ‖ equal to the Base EF. And so in equal Circles, equal Circumferences subtend equal Right Lines; *which was to be demonstrated.*

PROPOSITION XXX.

PROBLEM.

To cut a given Circumference into two equal Parts.

LET the given Circumference be ADB. It is required to cut the same into two equal Parts.

* 10. 1. Join AB, which bisect * in C; and let the Right Line CD be drawn from the Point C at Right Angles to AB; and join AD, DB.

Now because AC is equal to CB, and CD is common, the two Sides AC, CD, are equal to the two Sides BC, CD; but the Angle ACD is equal to the Angle BCD; for each of them is a Right Angle:
† 4. 1. Therefore the Base AD is † equal to the Base BD.
‡ 28 of this. But equal Right Lines cut ‡ off equal Circumferences. Wherefore the Circumference AD shall be equal to the Circumference BD. Therefore a given Circumference is cut into two equal Parts; *which was to be done.*

PRO-

PROPOSITION. XXXI.

THEOREM.

In a Circle, the Angle that is in a Semicircle, is a Right Angle; but the Angle in a greater Segment, is less than a Right Angle; and the Angle in a lesser Segment, greater than a Right Angle: Moreover, the Angle of a greater Segment, is greater than a Right Angle; and the Angle of a lesser Segment, is less than a Right Angle.

LET there be a Circle ABCD, whose Diameter is BC, and Center E; and join BA, AC, AD, DC. I say, the Angle which is in the Semicircle BAC is a Right Angle, that which is in the Segment ABC being greater than a Semicircle, *viz.* the Angle ABC, is less than a Right Angle; and that which is in the Segment ADC, being less than a Semicircle, that is, the Angle ADC is greater than a Right Angle.

For join AE, and produce BA to F.

Then because BE is equal to EA, the Angle EAB shall be * equal to the Angle EBA. And because AE is equal to EC, the Angle AEC will be † equal to the Angle CAE. Therefore the whole Angle BAC is * equal to the two Angles ABC, ACB; but the Angle FAC being without the Triangle ABC, is † equal to the two Angles ABC, ACB: Therefore the Angle BAC is equal to the Angle FAC; and so each of them is ‡ a Right Angle. Wherefore the Angle BAC in a Semicircle is a Right Angle. And because the two Angles ABC, BAC, of the Triangle ABC‡, are less than two Right Angles, and BAC is a Right Angle; then ABC is less than a Right Angle, and is in the Segment ABC greater than a Semicircle.

* 5. 1.
† 32. 1.
* 5. 1.
‡ Def. 10. 1.
‡ 17. 1.

And since ABCD is a quadrilateral Figure in a Circle, and the opposite Angles of any quadrilateral Figure described in a Circle, are ‖ equal to two Right Angles; the Angles ABC, ADC, are equal to two Right Angles, and the Angle ABC is less than a Right Angle: Therefore the remaining Angle ADC will

‖ 22 of this.

will be greater than a Right Angle, and is in the Segment ADC, which is less than a Semicircle.

I say, moreover, the Angle of the greater Segment contained under the Circumference A B C, and the Right Line AC, is greater than a Right Angle; and the Angle of the lesser Segment, contained under the Circumference ADC, and the Right Line AC is less than a Right Angle. This manifestly appears; for because the Angle contained under the Right Lines BA, AC, is a Right Angle, the Angle contained under the Circumference ABC, and the Right Line AC, will be greater than a Right Angle. Again, because the Angle contained under the Right Lines CA, AF, is a Right Angle, therefore the Angle which is contained under the Right Line AC, and the Circumference ADC, is less than a Right Angle. Therefore, *in a Circle, the Angle that is in a Semicircle, is a Right Angle; but the Angle in a greater Segment, is less than a Right Angle; and the Angle in a lesser Segment, greater than a Right Angle: Moreover, the Angle of a greater Segment, is greater than a Right Angle; and the Angle of a lesser Segment, is less than a Right Angle*; which was to be demonstrated.

PROPOSITION XXXII.

THEOREM.

If any Right Line touches a Circle, and a Right Line be drawn from the Point of Contact cutting the Circle; the Angles it makes with the Tangent Line, will be equal to those which are made in the alternate Segments of the Circle.

LET any Right Line EF touch the Circle ABCD in the Point B, and let the Right Line BD be any how drawn from the Point B cutting the Circle. I say, the Angles which BD makes with the Tangent Line EF, are equal to those in the alternate Segments of the Circle; that is, the Angle FBD is equal to an Angle made in the Segment DAB, *viz.* to the Angle DAB; and the Angle DBE equal to the Angle DCB, made in the Segment DCB. For

Draw

Draw * BA from the Point B at Right Angles to EF; and take any Point C in the Circumference BD, and join AD, DC, CB. * 11. 1.

Then because the Right Line EF touches the Circle ABCD in the Point B; and the Right Line BA is drawn from the Point of Contact B at Right Angles to the Tangent Line; the Center of the Circle ABCD, will † be in the Right Line BA; and so BA is a Diameter of the Circle, and the Angle ADB, in a Semicircle, is ‡ a Right Angle. Therefore the other Angles BAD, ABD, are equal to one Right Angle. But the Angle ABF is also a Right Angle: Therefore the Angle ABF, is * equal to the Angles BAD, ABD; and if ABD, which is common, be taken away, then the Angle DBF remaining, will be equal to that which is in the alternate Segment of the Circle, *viz.* equal to the Angle BAD. And because ABCD is a Quadrilateral Figure in a Circle, and the opposite Angles thereof are † equal to two Right Angles: The Angles DBF, DBE, will be equal to the Angles BAD, BCD. But BAD has been prov'd to be equal to DBF; therefore the Angle DBE, is equal to the Angle made in DCB, the alternate Segment of the Circle, *viz.* equal to the Angle DCB. Therefore, *if any Right Line touches a Circle, and a Right Line be drawn from the Point of Contact cutting the Circle; the Angles it makes with the Tangent Line, will be equal to those which are made in the alternate Segments of the Circle*; which was to be demonstrated.

† 19. 1. ‡ 31. of this. * 32. 1. † 22. of this.

PROPOSITION XXXIII.

PROBLEM.

To describe, upon a given Right Line, a Segment of a Circle, which shall contain an Angle, equal to a given Right-lin'd Angle.

LET the given Right Line be AB, and C the given Right-lin'd Angle. It is requir'd to describe the Segment of a Circle upon the given Right Line AB, containing an Angle, equal to the Angle C.

At

At the Point A, with the Right Line A B, make
‡ 23. 1. ‡ the Angle B A D equal to the Angle C, and draw
* 11. 1. * AE from the Point A, at the Right Angle, to A D.
† 10. 1. Likewise bisect † A B in F, and let F G be drawn from the Point F, at Right Angles, to A B, and join G B.

Then because A F is equal to F B, and F G is common, the two Sides A F, F G, are equal to the two Sides B F, F G; and the Angle A F G, is equal to the
‡ 4. of this. Angle B F G. Therefore the Base A G is ‡ equal to to the Base G B. And so if a Circle be describ'd about the Center G, with the Distance A G, this shall pass through the Point B. Describe the Circle, which let be A B E, and join E B. Now because A D is drawn from the Point A, the Extremity of the Diame-
* Cor. 16. of this. ter A E, at Right Angles to A E, the said A D will * touch the Circle. And since the Right Line A D touches the Circle A B E, and the Right Line A B, is drawn in the Circle from the Point of Contact A,
† 31. of this. the Angle D A B is † equal to the Angle made in the alternate Segment, *viz.* equal to the Angle A E B: But the Angle D A B, is equal to the Angle C. Therefore the Angle C will be equal to the Angle A E L. Wherefore the Segment of a Circle A E B is describ'd upon the given Right Line A B, containing an Angle A E B, equal to a given Angle C; *which was to be done.*

PROPOSITION XXXIV.

THEOREM

To cut off a Segment from a given Circle, that shall contain an Angle, equal to a given Right-lin'd Angle.

LET the given Circle be A B C, and the Right-lin'd Angle given D. It is requir'd to cut off a Segment from the Circle A B C, containing an Angle equal to the Angle D.

‡ 17. of this. Draw ‡ the Right Line E F, touching the Circle
* 23. 1. in the Point B, and make * the Angle F B C at the Point B equal to the Angle D.

Then because the Right Line E F touches the Circle A B C in the Point B, and B C is drawn from the

the Point of Contact B; the Angle F B C will be * equal to that in the alternate Segment of the Circle; but the Angle F B C is equal to the Angle D. Therefore the Angle in the Segment B A C, will be equal to the Angle D. Therefore the Segment B A C is cut off from the given Circle A B C, containing an Angle equal to the given Right-lin'd Angle D; *which was to be done.* * 32 of this.

PROPOSITION XXXV.

THEOREM.

If two Right Lines in a Circle mutually cut each other, the Rectangle contained under the Segments of the one, is equal to the Rectangle under the Segments of the other.

IN the Circle A B C D, let two Right Lines mutually cut each other in the Point E. I say the Rectangle contained under A E, and E C, is equal to the Rectangle contained under D E, E B.

If A D and D B pass thro' the Center, so that E be the Center of the Circle A B C D; it is manifest, since A E, E C, D E, E B, are equal; that the Rectangle, under A E, E C, is equal to the Rectangle under D E, E B.

But if A C, D B, do not pass thro' the Center, assume the Center of the Circle F; from which draw F G, F H, perpendicular to the Right Lines A C, D B, and join F B, F C, F E.

Then because the Right Line G F, drawn thro' the Center, cuts the Right Line A C, not drawn thro' the Center at Right Angles, it will also bisect * the same. Wherefore A G is equal to G C: And because the Right Line A C is cut into two equal Parts in the Point G, and into two unequal Parts in E, the Rectangle under A E, E C, together with the Square of E G, is † equal to the Square of G C. And if the common Square of G F be added, then the Rectangle under A E, E C, together with the Squares of E G, G F, is equal to the Squares of C G, G F. But the Square of F E is † equal to the Squares of E G, G F, and the Square of F C equal ‡ to the Squares of

* 4 of this.
† 5. 2.
‡ 47. 1.

of CG, GF. Therefore the Rectangle under AE, EC, together with the Square of FE, is equal to the Square of FC; but CF is equal to FB. Therefore the Rectangle under AE, EC, together with the Square of EF, is equal to the Square of FB. For the same Reason, the Rectangle under DE, EB, together with the Square of FE, is equal to the Square of FB. But it has been prov'd, that the Rectangle under AE, EC, together with the Square of FE, is also equal to the Square of FB. Therefore the Rectangle under AE, EC, together with the Square of FE, is equal to the Rectangle under DE, EB, together with the Square of FE. And if the common Square of FE be taken away, then there will remain the Rectangle under AE, EC, equal to the Rectangle under DE, EB. Wherefore, *if two Right Lines in a Circle mutually cut each other, the Rectangle contained under the Segments of the one, is equal to the Rectangle under the Segments of the other*; which was to be demonstrated.

PROPOSITION XXXVI.

THEOREM.

If some Point be taken without a Circle, and from that Point two Right Lines fall to the Circle, one of which cuts the Circle, and the other touches it; the Rectangle contain'd under the whole Secant Line, and its Part between the Convexity of the Circle and the assum'd Point, will be equal to the Square of the Tangent Line.

LET any Point D be assum'd without the Circle ABC, and let two Right Lines DCA, DB, fall from the said Point to the Circle; whereof DCA cuts the Circle, and DB touches it. I say the Rectangle under AD, DC, is equal to the Square of DB.

Now DCA either passes thro' the Center, or not. In the first place, let it pass thro' the Center of the Circle ABC, which let be E, and join EB. Then the Angle EBD is * a Right Angle. And so since the Right Line AC is bisected in E, and CD is added thereto, the Rectangle under AD, DC, together with

* 18 of this.

with the Square of E C, shall * be equal to the Square of E D. But E C is equal to E B; wherefore the Rectangle under A D, D C, together with the Square of E B, is equal to the Square of E D. But the Square of E D is † equal to the Square of E B, and B D. For the Angle E B D, is a Right Angle: Therefore the Rectangle under A D, D C, together with the Square of E B, is equal to the Squares of E B and B D; and if the common Square of E B be taken away, the Rectangle under A D, D C, remaining, will be equal to the Square of the Tangent Line B D.

* 6. 2.

† 47. 1.

Now let D C A not pass through the Center of the Circle A B C; and find ‡ the Center E thereof, and draw E F perpendicular to A C, and join E B, E C, E D. Therefore E F D is a Right Angle. And because a Right Line E F, drawn through the Center, cuts a Right Line A C at Right Angles, not drawn through the Center, it will * bisect the same at Right Angles; and so A F is equal to F C. Again, since the Right Line A C is bisected in F, and C D is added thereto, the Rectangle under A D, D C, together with the Square of F C, will be * equal to the Square of F D. And if the common Square of E F be added, then the Rectangle under A D, D C, together with the Squares of F C and F E, is † equal to the Squares of D F and F E. But the Square of D E, is equal to the Squares of D F and F E; for the Angle E F D is a Right one: And the Square of C E is ‡ equal to the Square of C F and F E. Therefore the Rectangle under A D, D C, together with the Square of C E, is equal to the Square of E D; but C E is equal to E B. Wherefore the Rectangle under A D, D C, together with the Square of E B, is equal to the Square of E D. But the Squares of E B and B D are * equal to the Square of E D; since the Angle E B D is a Right one. Wherefore the Rectangle under A D and D C, together with the Square of E B, is equal to the Squares of E B and B D. And if the common Square of E B be taken away, the Rectangle under A D and D C, remaining, will be equal to the Square of D B. Therefore, *if any Point be taken without a Circle, and from that Point two Right Lines fall to the Circle, one of which cuts the Circle, and the other touches it; the Rectangle contain'd under the whole Secant Line, and its*

‡ 1. of this.

* 3. of this.

its Part between the Convexity of the Circle and the assum'd Point, will be equal to the Square of the Tangent Line; which was to be demonstrated.

PROPOSITION XXXVII.

THEOREM.

If some Point be taken without a Circle, and two Right Lines be drawn from it to the Circle, so that one cuts it, and the other falls upon it; and if the Rectangle under the whole Secant Line, and the Part thereof, without the Circle, be equal to the Square of the Line falling upon the Circle, then this last Line will touch the Circle.

LET some Point D be assum'd without the Circle ABC, and from it draw two Right Lines DCA, DB, to the Circle, in such Manner that DCA cuts the Circle, and DA falls upon it: And let the Rectangle under AD, DC, be equal to the Square of DB. I say, the Right Line DB touches the Circle.

* 17. of this. For let the Right Line DE be drawn * touching the Circle ABC, and find F the Center of the Circle, and join EF, FB, FD.

† 18. 1. Then the Angle FED is † a Right Angle. And because DE touches the Circle ABC, and DCA cuts it, the Rectangle under AD, and DC, will be equal to the Square of DE. But the Rectangle under AD and DC, is ‡ equal to the Square of DB. ‡ By Hyp. Wherefore the Square of DE shall be equal to the Square of DB. And so the Line DE will be equal to the Line DB. But FE is equal to FB; Therefore the two Sides DE, EF, are equal to the two Sides DB, BF; and the Base FD is common. Wherefore the Angle DEF is equal to the Angle DBF; but DEF is a Right Angle; wherefore DBF is also a Right Angle, and FB produced is a Diameter. But a Right Line drawn at Right Angles, on the End of the Diameter of a Circle, touches the Circle; therefore BD necessarily touches the Circle. We prove this in the same Manner, if the Center of the Circle be in the Right Line CA. *If therefore any Point be assum'd without a Circle, and two Right*

2 *Lines*

Lines be drawn from it to the Circle, so that one cuts it, and the other falls upon it; and if the Rectangle under the whole Secant Line, and the Part thereof, without the Circle, be equal to the Square of the Line falling upon the Circle; then this last Line will touch the Circle; which was to be demonstrated.

Coroll. Hence, if from any Point without a Circle, several Right Lines A B, A C, are drawn cutting the Circle; the Rectangles comprehended under the whole Lines A B, A C, and their external Parts A E, A F, are equal between themselves. For if the Tangent A D be drawn, the Rectangle under B A and A E, is equal to the Square of A D; and the Rectangle under C A and A F, is equal to the same Square of A D: Therefore the Rectangles shall be equal.

The END *of the* THIRD BOOK.

EUCLID's ELEMENTS.

BOOK IV.

DEFINITIONS.

I. A *Right-lin'd Figure is ſaid to be inſcribed in a Right-lin'd Figure, when every one of the Angles of the inſcribed Figure, touches every one of the Sides of the Figure, wherein it is deſcribed.*

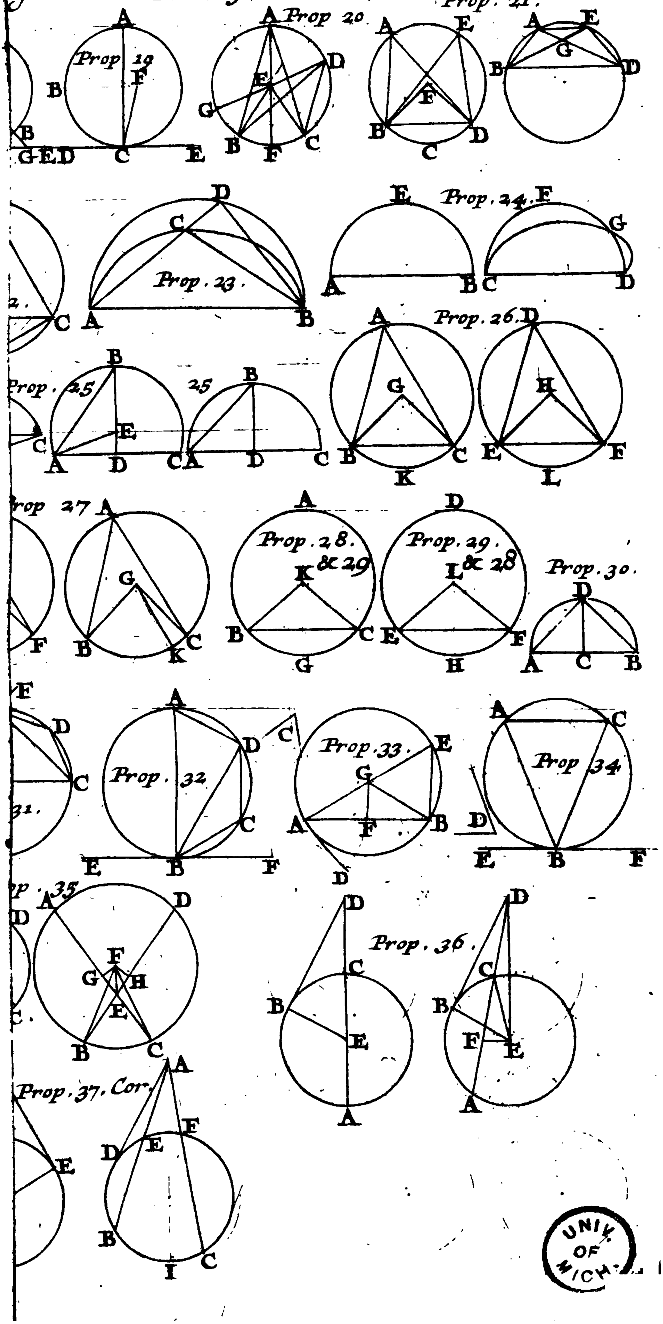
II. *In like Manner a Figure is ſaid to be deſcrib'd about a Figure, when every one of the Sides of the Figure, circumſcribed, touches every one of the Angles of the Figure about which it is circumſcribed.*

III. *A Right-lin'd Figure is ſaid to be inſcribed in a Circle, when every one of the Sides of that Figure which is inſcribed, touches the Circumference of the Circle.*

IV. *A Right-lin'd Figure is ſaid to be deſcribed about a Circle, when every one of the Sides of the circumſcribed Figure, touches the Circumference of the Circle.*

V. So

g. 100. Book 3. Plate 2.



V. *So likewise a Circle is said to be inscribed in a Right-lin'd Figure, when the Circumference of the Circle touches all the Sides of the Figure in which it is inscribed.*

VI. *A Circle is said to be described about a Figure, when the Circumference of the Circle touches all the Angles of the Figure which it circumscribes.*

VII. *A Right Line is said to be apply'd in a Circle, when its Extremes are in the Circumference of the Circle.*

PROPOSITION I.

PROBLEM.

To apply a Right Line in a given Circle, equal to a given Right Line, whose Length does not exceed the Diameter of the Circle.

LET the Circle given be ABC, and the given Right Line not greater than the Diameter be D. It is requir'd to apply a Right Line in the Circle ABC, equal to the Right Line D.

Draw BC the Diameter of the Circle; then if BC be equal to D, what was requir'd, is done; for in the Circle ABC there is applied the Right Line BC, equal to the Right Line D: But if not, the Diameter BC is greater than D, and put * CE equal to D; * 3. 1.
and about the Center C, with the Distance CE, let the Circle AEF be described; and join CA.

Then because the Point C is the Center of the Circle AEF, CA will be equal to CE; but D is equal to CE. Wherefore AC is equal to D. And so in the Circle ABC, there is applied a Right Line AC, equal to the given Right Line D, not greater than the Diameter; *which was to be done.*

PROPOSITION II.

PROBLEM.

In a given Circle, to describe a Triangle equiangular to a given Triangle.

LET ABC be a Circle given, and DEF a given Triangle. It is required to describe a Triangle in the Circle ABC, equiangular to the Triangle DEF.

* 17. 3. Draw the Right Line GAH touching * the Circle ABC in the Point A, and with the Right Line AH
† 23. 1. at the Point A, make † an Angle HAC, equal to the Angle DEF. Likewise at the same Point A, with the Line AG, make the Angle GAB equal to the Angle DFE, and join BC.

Then because the Right Line HAG touches the Circle ABC, and AC is drawn from the Point of Contact in the Circle; the Angle HAC shall be
‡ 32. 3. ‡ equal to ABC, the Angle in the alternate Segment of the Circle. But the Angle HAC is equal to the Angle DEF; therefore also the Angle ABC, is equal to the Angle DEF: For the same Reason, the Angle ACB is likewise equal to the Angle DFE. Where-
‖ Cor. 2. 32. 1. fore the other Angle BAC, shall be ‖ equal to the other Angle EDF. And consequently, the Triangle ABC is equiangular to the Triangle DEF, and is described in the Circle ABC; *which was to be done.*

PROPOSITION III.

PROBLEM.

About a given Circle to describe a Triangle, equiangular to a Triangle given.

LET ABC be the given Circle, and DEF the given Triangle. It is required to describe a Triangle about the Circle ABC equiangular to the Triangle DEF.

Produce the Side EF both Ways to the Points G and H, and find the Center of the Circle K, and any how draw the Line KB. Then at the Point K, with

KB

KB make * the Angle BKA equal to the Angle DEG; and the Angle BKC at the same Point K on the other Side the Line KB, equal to the Angle DFH; and thro' the Points A, B, C, let the Right Lines LAM, MBN, NCL, be drawn touching the Circle ABC. * 23. 1.

Then because the Lines LM, MN, NL, touch the Circle ABC in the Points A, B, C, and the Lines KA, KB, KC, are drawn from the Center K to the Points A, B, C; the Angles at the Points A, B, C, will be * Right Angles. And because the four Angles of the quadrilateral Figure AMBK are equal to four Right Angles, (for it may be divided into two Triangles,) and the Angles KAM, KBM, are each Right Angles; therefore the other Angles AKB, AMB are equal to two Right Angles. But DEG, DEF, are equal to two Right Angles; therefore the Angles AKB, AMB, are equal to the Angles DEG, DEF, whereof AKB is equal to DEG. Wherefore the other Angle AMB is equal to the other Angle DEF. In like Manner we demonstrate, that the Angle LNB is equal to the Angle DFE. Therefore the other Angle MLN is ‡ equal to the other Angle EDF. Wherefore the Triangle LNM is equiangular to the Triangle DEF, and is described about the Circle ABC; *which was to be done.* † 18. 3. ‡ Cor. 2. 32. 1.

PROPOSITION. IV.

PROBLEM.

To inscribe a Circle in a given Triangle.

LET ABC be a Triangle given. It is required to inscribe a Circle in the same.

Cut * the Angles ABC, BCA, into two equal Parts by the Right Lines BD, DC, meeting each other in the Point D. And from this Point draw DE, DF, DG, † perpendicular to the Sides AB, BC, AC. * 9. 1. † 12. 1.

Now because the Angle EBD is equal to the Angle FBD, and the Right Angle BED is equal to the Right Angle BFD; then the two Triangles EBD, DBF, have two Angles of the one, equal to two

Angles of the other, and one Side D B common to both, *viz.* that which subtends the equal Angles; therefore
‡ 26. 1. the other Sides of the one Triangle shall be ‡ equal to the other Sides of the other; and so D E shall be equal to D F. And for the same Reason, D G is equal to D F: Therefore D E is also equal to D G. And so the three Right Lines D E, D F, D G, are equal between themselves. Wherefore a Circle described about the Center D, with either of the Distances D E, D F, D G, will also pass thro' the other Points. And the Sides A B, B C, A C, will touch it; because the Angles at E, F, and G, are Right Angles. For if it should cut them, a Right Line drawn on the Extremity of the Diameter of a Circle at Right Angles, will fall with-
6. 3. in the Circle; which is * absurd. Therefore a Circle describ'd about the Center D, with either of the Distances D E, D F, D G, will not cut the Sides A B, B C, C A; wherefore it will touch them, and will be a Circle describ'd in the Triangle A B C. Therefore the Circle E F G is describ'd in the given Triangle A B C; *which was to be done.*

PROPOSITION V.

PROBLEM.

To describe a Circle about a given Triangle.

LET A B C be a given Triangle. It is requir'd to describe a Circle about the same.

* 30. 1. Bisect * the Sides A B, A C, in the Points D, E;
† 11. 1. from which Points let D F, E F, be drawn † at Right Angles to A B, A C, which will meet either within the Triangle A B C, or in the Side B C, or without the Triangle.

First let them meet in the Point F within the Triangle, and join B F, F C, F A. Then because A D is equal to D B, and D F is common, and at Right An-
‡ 4. 1. gles to A B; the Base A F will be ‡ equal to the Base F B. And after the same Manner we prove, that the Base C F is equal to the Base F A. Therefore also is B F equal to C F: And so the three Right Lines F A, F B, F C, are equal to each other. Wherefore a Circle describ'd about the Center F, with either of the Distances

Distances FA, FB, FC, will pass also thro' the other Points, and will be a Circle describ'd about the Triangle ABC. Therefore describe the Circle ABC.

Secondly, let DF, EF, meet each other in the Point F, in the Side BC, as in the second Figure, and join AF. Then we prove, as before, that the Point F is the Center of a Circle describ'd about the Triangle ABC.

Lastly, let the Right Lines DF, EF, meet one another again in the Point F, without the Triangle, as in the third Figure; and join AF, FB, FC. And because AD is equal to DB, and DF is common, and at Right Angles, the Base AF shall be equal to the Base BF. So likewise we prove, that CF is also equal to AF. Wherefore BF is equal to CF. And so again, if a Circle be describ'd on the Center F, with either of the Distances FA, FB, FC, it will pass thro' the other Points, and will be describ'd about the Triangle ABC; *which was to be done.*

Coroll. If a Triangle be Right-angl'd, the Center of the Circle falls in the Side opposite to the Right Angle; if acute-angl'd, it falls within the Triangle; and if obtuse-angl'd, it falls without the Triangle.

PROPOSITION VI.

PROBLEM.

To inscribe a Square in a given Circle.

LET ABCD be a Circle given. It is requir'd to inscribe a Square within the same.

Draw AC, BD, two Diameters of the Circle cutting one another at Right Angles, and join AB, BC, CD, DA.

Then because BE is equal to ED, (for E is the Center) and EA is common, and at Right Angles to BD, the Base BA shall be * equal to the Base AD; * 4. 1.
and for the same Reason BC, CD, as also BA, AD, are all equal to each other. Therefore the Quadrilateral Figure ABCD, is equilateral. I say it is also rectangular. For because the Right Line AB is a Diameter

Diameter of the Circle ABCD, BAD, will be a Semicircle. Wherefore the Angle BAD is * a Right Angle. And for the ſame Reaſon every of the Angles ABC, BCD, CDA, is a Right Angle. Therefore ABCD is a rectangular quadrilateral Figure: But it has alſo been prov'd to be equiangular. Wherefore it ſhall neceſſarily be a Square, and is deſcrib'd in the Circle ABCD; *which was to be done.*

* 31. 1.

PROPOSITION VII.

PROBLEM.

To deſcribe a Square about a given Circle.

LET ABCD be a Circle given. It is requir'd to deſcribe a Square about the ſame.

Draw AC, BD, two Diameters of the Circle cutting each other at Right Angles, and thro' the Points A, B, C, D, draw * FG, GH, HK, KF, Tangents to the Circle ABCD.

* 17. 3.

Then becauſe FG touches the Circle ABCD, and EA is drawn from the Center E to the Point of Contact A, the Angles at A will be † Right Angles. For the ſame Reaſon, the Angles at the Points B, C, D, are Right Angles. And ſince the Angle AEB is a Right Angle, as alſo EBG, GH ſhall be ‡ parallel to AC; and for the ſame Reaſon, AC to KF. In this Manner we prove likewiſe, that GF and HK are parallel to BED; and ſo GF is parallel to HK. Therefore GK, GC, AK, FB, BK, are Parallelograms; and ſo GF is * equal to HK, and GH to FK. And ſince AC is equal to BD, and AC * equal to either GH, or FK, and BD equal to either GF, or HK; GH, or FK, is equal to GF, or HK. Therefore FGHK is an equilateral quadrilateral Figure: I ſay it is alſo equiangular. For becauſe GBEA is a Parallelogram, and AEB is a Right Angle, then AGB ſhall be alſo a Right Angle. In like manner we demonſtrate, that the Angles at the Points H, K, F, are Right Angles. Therefore the quadrilateral Figure FGHK is rectangular; but it has been prov'd to be equilateral likewiſe. Wherefore it muſt neceſſarily be a Square, and is deſcrib'd about the Circle ABCD; *which was to be done.*

† 18. 3.

‡ 28. 1.

* 34. 1.

PRO-

PROPOSITION VIII.

PROBLEM.

To describe a Circle in a given Square.

LET the given Square be ABCD. It is requir'd to describe a Circle within the same.

Bisect * the Sides AB, AD, in the Points F, E; and draw † EH thro' E, parallel to AB, or DC; and FK thro' F, parallel † to BC, or AD. Then AK, KB, AH, HD, AG, GC, BG, GD, are all Parallelograms, and their opposite Sides are ‡ equal. And because DA is equal to AB, and AE is half of AD, and AF half of AB, AE shall be equal to AF; but the opposite Sides are also equal. Therefore FG is equal to GE. In like manner we demonstrate, that GH, or GK, is equal to either FG, or GE. Therefore GE, GF, GH, GK, are equal to each other: And so a Circle being describ'd about the Center G, with either of the Distances GE, GF, GH, GK, will also pass thro' the other Points, and shall touch the Sides AB, BC, CD, DA, because the Angles at E, F, H, K, are Right Angles. For if the Circle should cut the Sides of the Square, a Right Line, drawn from the End of the Diameter of a Circle at Right Angles, will fall within the Circle; which is * absurd. Wherefore a Circle describ'd about the Center G, with either of the Distances GE, GF, GH, GK, will not cut AB, BC, CD, DA, the Sides of the Square. Wherefore it shall necessarily touch them, and will be describ'd in the Square ABCD; *which was to be done.*

* 10. 1.
† 31. 1.
‡ 34. 1.
* 16. 3.

PROPOSITION. IX.

PROBLEM.

To describe a Circle about a Square given.

LET ABCD be a Square given. It is requir'd to circumscribe a Circle about the same.

Join AC, BD, mutually cutting one another in the Point E,

And ſince DA is equal to AB, and AC is common, the two Sides DA, AC, are equal to the two Sides BA, AC; but the Baſe DC is equal to the Baſe BC. Therefore the Angle DAC will * be equal to the Angle BAC: And conſequently the Angle DAB is biſected by the Right Line AC. In the ſame Manner we prove, that each of the Angles ABC, BCD, CDA, are biſected by the Right Lines AC, DB.

* 8. 1.

Then becauſe the Angle DAB is equal to the Angle ABC, and the Angle EAB is half of the Angle DAB, and the Angle EBA half of the Angle ABC; the Angle EAB ſhall be equal to the Angle EBA: And ſo the Side EA is † equal to the Side EB. In like manner we demonſtrate, that each of the Right Lines EC, ED, is equal to each of the Right Lines EA, EB. Therefore the four Right Lines EA, EB, EC, ED, are equal between themſelves. Wherefore a Circle being deſcrib'd about the Center E, with either of the Diſtances EB, EC, ED, will alſo paſs thro' the other Points, and will be deſcrib'd about the Square ABCD; *which was to be done.*

† 6. 1.

PROPOSITION X.

PROBLEM.

To make an Iſoſceles Triangle, having each of the Angles at the Baſe double to the other Angle.

CUT * any given Right Line AB in the Point C, ſo that the Rectangle contain'd under AB, BC, be equal to the Square of AC; then about the Center A, with the Diſtance AB, let the Circle BDE be deſcrib'd; and † in the Circle BDE apply † the Right Line BD equal to AC; which is not greater than the Diameter. This being done, join DA, DC, and deſcribe ‡ a Circle ACD about the Triangle ADC.

* 11. 2. † 1 *of this.* ‡ 5 *of this.*

Then becauſe the Rectangle ABC is equal to the Square of AC, and AC is equal to BD, the Rectangle under AB, BC, ſhall be equal to the Square of BD. And becauſe ſome Point B is taken without the Circle ACD, and from that Point there fall two Right Lines BCA, BD, to the Circle, one of which

cuts the Circle, and the other falls on it. And since the Rectangle under AB, BC, is equal to the Square of BD, the Right Line BD shall * touch the Circle ACD. And since BD touches it, and DC is drawn from the Point of Contact D, the Angle BDC is equal to the Angle in the alternate Segment of the Circle, *viz.* equal † to the Angle DAC. And since the Angle BDC is equal to the Angle DAC; if CDA, which is common, be added, the whole Angle BDA is equal to the two Angles CDA, DAC. But the outward Angle BCD is ‡ equal to CDA, DAC. Therefore BDA is equal to BCD. But the Angle BDA * is equal to the Angle CBD, because the Side AD is equal to the Side AB. Wherefore DBA shall be equal to BCD: And so three Angles BDA, DBA, BCD, are equal to each other. And since the Angle DBC is equal to the Angle BCD, the Side BD is † equal to the Side DC. But BD is put equal to CA. Therefore CA is equal to CD. And so the Angle CDA is equal to the Angle DAC. Therefore the Angles CDA, DAC, taken together, are double to the Angle DAC. But the Angle BCD is equal to the Angles CDA, DAC. Therefore the Angle BCD is double to the Angle DAC. But BCD is equal to BDA, or DBA. Wherefore BDA, or DBA, is double to DAB. Therefore the Isosceles Triangle ADB is made, having one of the Angles at the Base, double to the other Angle; *which was to be done.*

* 37. 3.
† 32. 3.
‡ 32. 1.
* 5. 1.
† 6. 1.

PROPOSITION XI.

PROBLEM.

To describe an equilateral and equiangular Pentagon in a given Circle.

LET ABCDE be a Circle given. It is requir'd to describe an equilateral and equiangular Pentagon in the same.

Make an Isosceles Triangle FGH, having * each of the Angles at the Base GH, double to the other Angle F; and describe the Triangle ADC in the Circle ABCDE, equiangular † to the Triangle FGH; so that the

* 10 of this.
† 2 of this.

the Angle CAD be equal to that at F, and ACD, CDA, each equal to the Angles G or H. Wherefore the Angles ACD, CDA, are each double to the Angle CAD. This being done, bisect * ACD, CDA, by the Right Lines CE, DB, and join AB, BC, DE, EA.

* 9. 1.

Then because each of the Angles ACD, CDA, is double to CAD, and they are bisected by the Right Lines CE, DB; the five Angles DAC, ACE, ECD, CDB, BDA, are equal to each other. But equal Angles stand * upon equal Circumferences. Therefore the five Circumferences AB, BC, CD, DE, EA, are equal to each other. But equal Circumferences subtend † equal Right Lines. Therefore the five Right Lines AB, BC, CD, DE, EA, are equal to each other. Wherefore ABCDE is an equilateral Pentagon. I say, it is also equiangular; for because the Circumference AB is equal to the Circumference DE, by adding the Circumference BCD, which is common, the whole Circumference ABCD is equal to the whole Circumference EDCB, but the Angle AED stands on the Circumference ABCD, and BAE on the Circumference EDCB: Therefore the Angle BAE is equal to the Angle AED. For the same Reason, each of the Angles ABC, BCD, CDE, is equal to BAE, or AED. Wherefore the Pentagon ABCDE is equiangular; but it has been prov'd to be also equilateral. And consequently there is an equilateral and equiangular Pentagon inscribed in a given Circle; *which was to be done.*

* 26. 3.
† 29. 3.

PROPOSITION XII.

PROBLEM.

To describe an equilateral and equiangular Pentagon about a Circle given.

LET ABCDE be the given Circle. It is required to describe an equilateral and equiangular Triangle about the same.

Let A, B, C, D, E, be the angular Points of a Pentagon supposed to be inscribed * in the Circle; so that the Circumferences AB, BC, CD, DE, EA, be equal;

* By 11 of this.

equal; and let the Right Lines GH, HK, KL, LM, MG, be drawn touching † the Circle in the Points A, B, C, D, E: Let F be the Center of the Circle ABCDE, and join FB, FK, FC, FL, FD. † 17. 3.

Then because the Right Line KL touches the Circle ABCDE in the Point C, and the Right Line FC is drawn from the Center F to C, the Point of Contact; FC will be ‡ perpendicular to KL: And so both the Angles at C are Right Angles. For the same Reason, the Angles at the Points B, D, are Right Angles. And because FCK is a Right Angle, the Square of FK will be * equal to the Squares of FC, CK: And for the same Reason, the Square of FK is equal to the Squares of FB, BK. Therefore the Squares of FC, CK are equal to the Squares of FB, BK. But the Square of FC is equal to the Square of FB. Wherefore the Square of CK shall be equal to the Square BK; and so BK is equal to CK. And because FB is equal to FC, and FK is common; the two Sides BF, FK, are equal to the two CF, FK, and the Base BK is equal to the Base KC; and so the Angle BFK shall be † equal to the Angle KFC, and the Angle BKF to the Angle FKC. Therefore the Angle BFC is double to the Angle KFC, and the Angle BKC double to the Angle FKC: For the same Reason, the Angle CFD is double to the Angle CFL, and the Angle CLD double to the Angle CLF. And because the Circumference BC is equal to the Circumference CD, the Angle BFC shall be ‡ equal to the Angle CFD. But the Angle BFC is double to the Angle KFC, and the Angle DFC double to LFC. Therefore the Angle KFC is equal to the Angle CFL. And so FKC, FLC, are two Triangles having two Angles of the one equal to two Angles of the other, each to each, and one Side of the one equal to one Side of the other, *viz.* the common Side FC; wherefore they shall have ‖ the other Sides of the one equal to the other Sides of the other; and the other Angle of the one, equal to the other Angle of the other. Therefore the Right Line KC is equal to the Right Line CL, and the Angle FKC to the Angle FLC. And since KC is equal to CL, KL shall be double to KC. And by the same Reason, we prove that HK

‡ 18. 3.

* 47. 1.

† 5. 1.

‡ 27. 3.

‖ 26. 1.

HK is double to BK. Again, becauſe BK has been prov'd equal to KC, and KL the double to KC, as alſo HK the double of BK, HK ſhall be equal to KL. So likewiſe, we prove that GH, GM, and ML, are each equal to HK, or KL. Therefore the Pentagon GHKLM is equilateral. I ſay alſo, it is equiangular; for becauſe the Angle FKC is equal to the Angle FLC; and the Angle HKL has been prov'd to be double to the Angle FKC; and alſo KLM double to FLC: Therefore the Angle HKL ſhall be equal to the Angle KLM. By the ſame Reaſon we demonſtrate, that every of the Angles KHG, HGM, GML, is equal to the Angle HKL, or KLM. Therefore the five Angles GHK, HKL, KLM, LMG, MGH, are equal between themſelves. And ſo the Pentagon GHKLM is equiangular, and it has been proved likewiſe to be equilateral, and deſcribed about the Circle ABCDE; *which was to be done.*

PROPOSITION XIII.

PROBLEM.

To inſcribe a Circle in an equilateral and equiangular Pentagon.

LET ABCDE be an equilateral and equiangular Pentagon. It is required to inſcribe a Circle in the ſame.

* 9. 1. Biſect * the Angles BCD, CDE, by the Right Lines CF, DF, and from the Point F wherein CF, DF, meet each other, let the Right Lines FB, FA, FE, be drawn. Now becauſe BC is equal to CD, and CF is common, the two Sides BC, CF, are equal to the two Sides DC, CF; and the Angle BCF is equal to the Angle DCF. Therefore the
† 4. 1. Baſe BF is † equal to the Baſe FD; and the Triangle BFC equal to the Triangle DCF, and the other Angles of the one equal to the other Angles of the other, which are ſubtended by the equal Sides: Therefore the Angle CBF ſhall be equal to the Angle CDF. And becauſe the Angle CDE is double to the Angle CDF, and the Angle CDE is equal to the

Angle A B C, as also C D F equal to C B F; the Angle C B A will be double to the Angle C B F; and so the Angle A B F equal to the Angle C B F. Wherefore the Angle A B C is bisected by the Right Line B F. After the same Manner we prove, that either of the Angles B A E or A E D is bisected by the Right Lines A F, F E. From the Point F draw * F G, F H, F K, F L, F M, perpendicular to the Right Lines A B, B C, C D, D E, E A. Then since the Angle H C F is equal to the Angle K C F; and the Right Angle F H C equal to the Right Angle F K C; the two Triangles F H C, F K C shall have two Angles of the one equal to two Angles of the other, and one Side of the one equal to one Side of the other, *viz.* the Side F C common to each of them: And so the other Sides of the one will be † equal to the other Sides of the other: And the Perpendicular F H equal to the Perpendicular F K. In the same Manner we demonstrate, that F L, F M, or F G, is equal to F H, or F K. Therefore the five Right Lines F G, F H, F K, F L, F M, are equal to each other. And so a Circle described on the Center F, with either of the Distances F G, F H, F K, F L, F M, will pass thro' the other Points, and shall touch the Right Lines A B, B C, C D, D E, E A; since the Angles at G, H, K, L, M, are Right Angles: For if it does not touch them, but cuts them, a Right Line drawn from the Extremity of the Diameter of a Circle at Right Angles to the Diameter, will fall within the Circle; which is ‡ absurd. Therefore a Circle described on the Center F with the Distance of any one of the Points G, H, K, L, M, will not cut the Right Lines A B, B C, C D, D E, E A; and so will necessarily touch them; *which was to be done.*

* 12. 1.
† 26. 1.
‡ 16. 3.

Coroll. If two of the nearest Angles of an eqilateral and equiangular Figure be bisected, and from the Point in which the Lines bisecting the Angles meet, there be drawn Right Lines to the other Angles of the Figure, all the Angles of the Figure will be bisected.

PROPOSITION XIV.

PROBLEM.

To describe a Circle about a given equilateral and equiangular Pentagon.

LET ABCDE be an equilateral and equiangular Pentagon. It is requir'd to describe a Circle about the same.

Bisect both the Angles BCD, CDE, by the Right Lines CF, FD, and draw FB, FA, FE, from the Point F, in which they meet. Then each of the Angles CBA, BAE, AED, shall be bisected * by the Right Lines BF, FA, FE. And since the Angle BCD, is equal to the Angle CDE; and the Angle FCD, is half the Angle BCD, as likewise CDF, half CDE; the Angle FCD, will be equal to the Angle FDC; and so the Side CF †, equal to the Side FD. We demonstrate in like Manner, that FB, FA, or FE, is equal to FC, or FD. Therefore the five Right Lines, FA, FB, FC, FD, FE, are equal to each other. And so a Circle being describ'd on the Center F, with any of the Distances FA, FB, FC, FD, FE, will pass thro' the other Points, and will be describ'd about the equilateral and equiangular Pentagon ABCDE; *which was to be done.*

* *Cor. of Preced.*

† 6. 1.

PROPOSITION XV.

PROBLEM.

To inscribe an equilateral and equiangular Hexagon in a given Circle.

LET ABCDEF be a Circle given. It is requir'd to inscribe an equilateral and equiangular Hexagon therein.

Draw AD a Diameter of the Circle ABCDEF, and let G be the Center; and about the Point D, as a Center, with the Distance DG, let a Circle EGCH, be describ'd; join EG, GC, which produce to the Points B, F: Likewise join AB, BC, CD, DE, EF, FA.

FA. I say ABCDEF is an equilateral and equiangular Hexagon.

For since the Point G is the Center of the Circle ABCDEF, GE will be equal to GD. Again, because the Point D is the Center of the Circle EGCH, DE shall be equal to DG: But GE has been prov'd equal to GD. Therefore GE is equal to ED. And so EGD is an equilateral Triangle; and consequently the three Angles thereof, EGD, GDE, DEG, are * equal between themselves: But the three Angles of a Triangle are † equal to two Right Angles. Therefore the Angle EGD, is a third Part of two Right Angles. In the same Manner we demonstrate, that DGC is one third Part of two Right Angles: And since the Right Line CG, standing upon the Right Line EB, makes ‡ the adjacent Angles EGC, CGB; the other Angle CGB, is also one third Part of two Right Angles. Therefore the Angles EGD, DGC, CGB, are equal between themselves: And the Angles that are vertical to them, *viz.* the Angles BGA, AGF, FGE, are * equal to the Angles EGD, DGC, CGB. Wherefore the six Angles EGD, DGC, CGB, BGA, AGF, FGE, are equal to one another. But equal Angles stand † on equal Circumferences. Therefore six Circumferences, AB, BC, CD, DE, EF, FA, are equal to each other. But equal Right Lines subtend ‡ equal Circumferences. Therefore the six Right Lines are equal between themselves; and accordingly the Hexagon ABCDEF is equilateral: I say it is also equiangular. For, because the Circumference AF is equal to the Circumference ED, add the common Circumference ABCD, and the whole Circumference FABCD, is equal to the whole Circumference EDCBA. But the Angle FED, stands on the Circumference FABCD; and the Angle AFE, on the Circumference EDCBA. Therefore the Angle AFE is * equal to the Angle DEF. In the same Manner we prove, that the other Angles of the Hexagon ABCDEF, are severally equal to AFE, or FED. Therefore the Hexagon ABCDEF is equiangular. But it has been prov'd to be also equilateral, and is inscrib'd in the Circle ABCDEF; *which was to be done.*

* *Cor.* 5. 1.
† 32. 1.
‡ 13. 1.
* 18. 1.
† 26. 3.
‡ 29. 3.
* 27. 3.

Coroll. From hence it is manifest, that the Side of the Hexagon is equal to the Semidiameter of the Circle.

And if we draw thro' the Points A, B, C, D, E, F, Tangents to the Circle, an equilateral and equiangular Hexagon will be describ'd about the Circle, as is manifest from what has been said concerning the Pentagon. And so likewise may a Circle be inscrib'd and circumscrib'd about a given Hexagon; *which was to be done.*

PROPOSITION XVI.

PROBLEM.

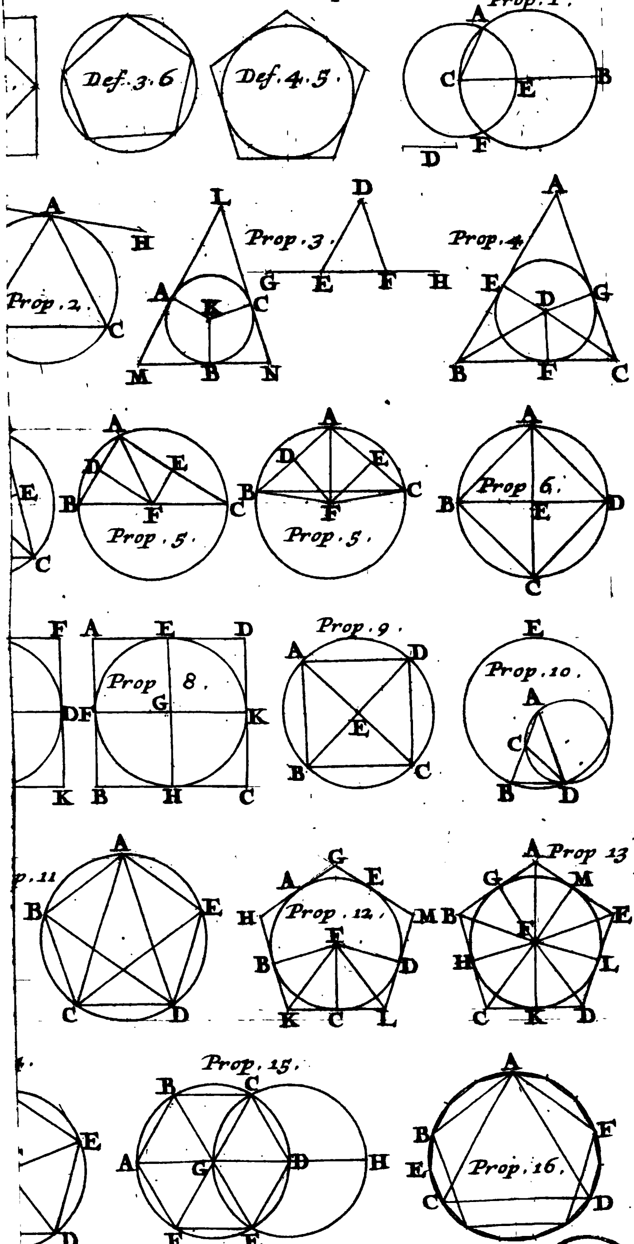
To describe an equilateral and equiangular Quindecagon in a given Circle.

LET ABCD be a Circle given. It is requir'd to describe an equilateral and equiangular Quindecagon in the same.

Let AC be the Side of an equilateral Triangle inscrib'd in the Circle ABCD, and AB the Side of a Pentagon. Now if the whole Circumference of the Circle ABCD be divided into fifteen equal Parts, the Circumference ABC, one third of the whole, shall be five of the said fifteen equal Parts; and the Circumference AB, one Fifth of the Whole will be three of the said Parts. Wherefore the remaining Circumference BC, will be two of the said Parts. And if BC be bisected in the Point E, BE, or EC, will be one fifteenth Part of the whole Circumference ABCD. And so if BE, EC, be joined, and either EC, or EB, be continually apply'd in the Circle, there shall be an equilateral and equiangular Quindecagon describ'd in the Circle ABCD; *which was to be done.*

If, according to what has been said of the Pentagon, Right Lines are drawn thro' the Divisions of the Circle touching the same, there will be describ'd about the Circle an equilateral and equiangular Quindecagon. And, moreover, a Circle may be inscrib'd, or circumscrib'd, about a given equilateral and equiangular. Quindecagon.

EUCLID's



Def. 3. 6
Def. 4. 5
Prop. 1
Prop. 2
Prop. 3
Prop. 4
Prop. 5
Prop. 5
Prop 6
Prop 8
Prop. 9
Prop. 10
p. 11
Prop. 12
Prop 13
Prop. 15
Prop. 16

EUCLID's ELEMENTS.

BOOK V.

DEFINITIONS.

I. *A PART, is a Magnitude of a Magnitude, the less of the greater, when the lesser measures the greater.*

II. *But a Multiple is a Magnitude of a Magnitude, the greater of the lesser, when the lesser measures the greater.*

III. *Ratio, is a certain mutual Habitude of the Magnitudes of the same kind, according to Quantity.*

IV. *Magnitudes are said to have Proportion to each other, which being multiplied can exceed one another.*

V. *Magnitudes are said to be in the same Ratio, the first to the second, and the third to the fourth, when the Equimultiples of the first and third, compared with the Equimultiples of the second and fourth, according to any Multiplication whatsoever, are either both together greater, equal, or less, than the Equi-*

multiples of the ſecond and fourth, if thoſe be taken that anſwer each other.

That is, if there be four Magnitudes, and you take any Equimultiples of the firſt and third, and alſo any Equimultiples of the ſecond and fourth. And if the Multiple of the firſt be greater than the Multiple of the ſecond, and alſo the Multiple of the third greater than the Multiple of the fourth: Or, if the Multiple of the firſt be equal to the Multiple of the ſecond; and alſo the Multiple of the third equal to the Multiple of the fourth: Or, laſtly, if the Multiple of the firſt be leſs than the Multiple of the ſecond; and alſo that of the third leſs than that of the fourth; and theſe Things happen according to every Multiplication whatſoever; then the four Magnitudes are in the ſame Ratio, the firſt to the ſecond, as the third to the fourth.

VI. *Magnitudes that have the ſame Proportion, are called Proportionals.*

Expounders uſually lay down here that Definition which *Euclid* has given for Numbers only, in his ſeventh Book, *viz.* That

Magnitudes are ſaid to be Proportionals, when the firſt is the ſame Equimultiple of the ſecond, as the third is of the fourth, or the ſame Part, or Parts.

But this Definition appertains only to Numbers and Commenſurable Quantities; and ſo ſince it is not univerſal, *Euclid* did well to reject it in this Element, which treats of the Properties of all Proportionals; and to ſubſtitute another general one, agreeing to all Kinds of Magnitudes. In the mean time, Expounders very much endeavour to demonſtrate the Definition here laid down by *Euclid*, by the uſual received Definition of Proportional Numbers; but this much eaſier flows from that, than that from this; which may be thus demonſtrated:

Firſt, Let A, B, C, D, be four Magnitudes which are in the ſame Ratio, according to the Conditions that Magnitudes in the ſame Ratio muſt have, laid down in the fifth Definition. And let the firſt be a Multiple of the ſecond. I ſay, the third is alſo the

ſame

ſame Multiple of the fourth. For Example: Let A be equal to 5 B. Then C ſhall be equal to 5 D. Take any Number. For Example, 2, by which let 5 be multiplied, and the Product will be 10: And let 2 A, 2 C, be Equimultiples of the firſt and third Magnitudes A and C:

A : B :: C : D

2 A, 10 B, 2 C, 10 D

Alſo, let 10 B and 10 D be Equimultiples of the ſecond and fourth Magnitudes B and D. Then (by *Def.* 5.) if 2 A be equal to 10 B, 2 C ſhall be equal to 10 D. But ſince A (from the *Hypotheſis*) is five Times B, 2 A ſhall be equal to 10 B; and ſo 2 C equal to 10 D, and C equal to 5 D; that is, C will be five Times D. W. W. D.

Secondly, Let A be any Part of B; then C will be the ſame Part of D. For becauſe A is to B, as C is to D, and ſince A is ſome Part of B; then B will be a Multiple of A: And ſo (by *Caſe* 1.) D will be the ſame Multiple of C, and accordingly C ſhall be the ſame Part of the Magnitude D, as A is of B. W. W. D.

Thirdly, Let A be equal to any Number of whatſoever Parts of B. I ſay, C is equal to the ſame Number of the like Parts of D. For Example: Let A be a Fourth Part of five Times B; that is, let A be equal to $\frac{5}{4}$ B. I ſay, C is alſo equal to $\frac{5}{4}$ D. For becauſe A is equal to $\frac{5}{4}$ B, each of them being multiplied by 4, then 4 A will be equal to 5 B. And ſo if the Equimultiples of the firſt and third, *viz.* 4 A, 4 C be aſſum'd; as alſo the Equimultiples of the ſecond and fourth, *viz.* 5 B, 5 D, and (by the *Definition*) if 4 A is equal to 5 B; then 4 C is equal to 5 D. But 4 A has been prov'd equal to 5 B, and ſo 4 C ſhall be equal to 5 D, and C equal to $\frac{5}{4}$ D. W. W. D.

A : B :: C : D.

4 A, 5 B, 4 C, 5 D.

And univerſally, If A be equal to $\frac{n}{m}$ B, C will be equal to $\frac{n}{m}$ D. For let A and C be multiplied by m, and B and D by n. And becauſe A is equal to $\frac{n}{m}$ B; m A ſhall be equal to n B; wherefore (by *Def.* 5) m C will be equal to n D and C equal to $\frac{n}{m}$ D. W. W. D.

A : B :: C : D

mA, nB, mC, nD

VII. *When of Equimultiples, the Multiple of the First exceeds the Multiple of the Second, but the Multiple of the Third does not exceed the Multiple of the Fourth; then the First to the Second is said to have a greater Proportion than the Third to the Fourth.*

VIII. *Analogy is a Similitude of Proportions.*

IX. *Analogy at least consists of three Terms.*

X. *When three Magnitudes are Proportionals, the First is said to have to the Third, a Duplicate Ratio to what it has to the Second.*

XI. *But when four Magnitudes are Proportionals, the First shall have a Trilicate Ratio to the Fourth of what it has to the Second; and so always one more in Order, as the Proportionals shall be extended.*

XII. *Homologous Magnitudes, or Magnitudes of a like Ratio, are said to be such whose Antecedents are to the Antecedents, and Consequents to the Consequents.*

XIII. *Alternate Ratio, is the comparing of the Antecedent with the Antecedent, and the Consequent with the Consequent.*

XIV. *Inverse Ratio, is when the Consequent is taken as the Antecedent, and so compared with the Antecedent as a Consequent.*

XV. *Compounded Ratio, is when the Antecedent and Consequent taken both as one, is compared to the Consequent itself.*

XVI. *Divided Ratio, is when the Excess wherein the Antecedent exceeds the Consequent, is compared with the Consequent.*

XVII. *Converse Ratio, is when the Antecedent is compared with the Excess, by which the Antecedent exceeds the Consequent.*

XVIII. *Ratio of Equality, is where there are taken more than two Magnitudes in one Order, and a like Number of Magnitudes in another Order, comparing two to two being in the same Proportion; and it shall be in the first Order of Magnitude, as the First is to the Last, so in the second Order of Magnitudes is the First to the Last: Or otherwise, it is the Comparison of the Extremes together, the Means being omitted.*

XIX. *Or-*

XIX. *Ordinate Proportion, is when, as the Antecedent is to the Consequent, so is the Antecedent to the Consequent; and as the Consequent is to any other, so is the Consequent to any other.*

XX. *Perturbate Proportion, is when there are three Magnitudes, and others also, that are equal to these in Multitude as in the first Magnitudes the Antecedent is to the Consequent; so in the second Magnitudes is the Antecedent, to the Consequent: And as in the first Magnitudes the Consequent is to some other, so in the second Magnitudes, is some other to the Antecedent.*

AXIOMS.

I. *EQUIMULTIPLES of the same, or of equal Magnitudes, are equal to each other.*

II. *Those Magnitudes that have the same Equimultiple, or whose Equimultiples are equal, are equal to each other.*

PRO-

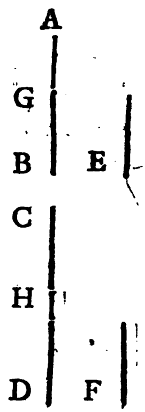
PROPOSITION I.

THEOREM.

If there be any Number of Magnitudes Equimultiples of a like Number of Magnitudes, each to each; whatsoever Multiple any one of the former Magnitudes is of its Correspondent one, the same Multiple is all the former Magnitudes of, all the latter.

LET there be any Number of Magnitudes AB, CD, Equimultiples of a like Number of Magnitudes E, F, each of each. I say, what Multiple the Magnitude AB is of E, the same Multiple AB, and CD, together, is of E and F together.

For because AB and CD are Equimultiples of E and F, as many Magnitudes equal to E, that are in AB, so many shall be equal to F in CD. Now divide AB into Parts equal to E, which let be AG, GB, and CD into Parts equal to F, *viz.* CH, HD. Then the Multitude of Parts, CH, HD, shall be equal to the Multitude of Parts AG, GB. And since AG is equal to E, and CH to F; AG and CH, together, shall be equal to E and F together. By the same Reason, because GB is equal to E, and HD to F, GB and HD will be equal to E and F together. Therefore, as often as E is contain'd in AB, so often is E and F contain'd in AB and CD. And so, as often as E is contain'd in AB, so often are E and F, together, contain'd in AB and CD together. Therefore, *if there are any Number of Magnitudes Equimultiples of a like Number of Magnitudes, each to each; whatsoever Multiple any one of the former Magnitudes is of its Correspondent one, the same Multiple is all the former Magnitudes of, all the latter*; which was to be demonstrated.



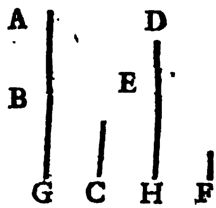
PRO-

PROPOSITION II.

THEOREM.

If the First be the same Multiple of the Second, as the Third is of the Fourth; and if the Fifth be the same Multiple of the Second, as the Sixth is of the Fourth; then shall the First, added to the Fifth, be the same Multiple of the Second, as the Third, added to the Sixth, is of the Fourth.

LET the first AB be the same Multiple of the second C, as the third DE is of the fourth F; and let the fifth BG be the same Multiple of the second C, as the sixth EH is of the fourth F. I say the first added to the fifth, *viz.* AG, is the same Multiple of the second C, as the third added to the sixth, *viz.* DH, is of the fourth F.



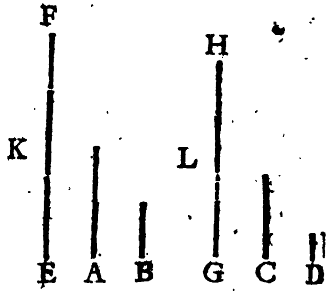
For because AB is the same Multiple of C, as DE is of F, there are as many Magnitudes equal to C in AB, as there are Magnitudes equal to F in DE. And for the same Reason there are as many Magnitudes equal to C in BG, as there are Magnitudes equal to F in EH. Therefore there are as many Magnitudes equal to C, in the whole AG, as there are Magnitudes equal to F in DH. Wherefore AG is the same Multiple of C, as DH is of F. And so the first added to the fifth AG, is the same Multiple of the second C, as the third, added to the sixth DH, is of the fourth F. Therefore, *if the First be the same Multiple of the Second, as the Third is of the Fourth; and if the Fifth be the same Multiple of the Second, as the Sixth is of the Fourth; then shall the First, added to the Fifth, be the same Multiple of the Second, as the Third, added to the Sixth, is of the Fourth*; which was to be demonstrated.

PROPOSITION. III.

THEOREM.

If the First be the same Multiple of the Second, as the Third is of the Fourth, and there be taken Equimultiples of the First and Third; then will each of the Magnitudes taken be Equimultiples of the Second and Fourth.

LET the first A be the same Multiple of the second B, as the third C is of the fourth D; and let EF, GH, be Equimultiples of A and C. I say EF is the same Multiple of B, as GH is of D.



For because EF is the same Multiple of A, as GH is of C, there are as many Magnitudes equal to A in EF, as there are Magnitudes equal to C in GH. Now divide EF into the Magnitudes EK, KF, equal to A, and GH into the Magnitudes GL, LH, equal to C. Then the Number of the Magnitudes EK, KF, will be equal to the Number of the Magnitudes GL, LH. And because A is the same Multiple of B, as C is of D, and EK is equal to A, and GL to C; EK will be the same Multiple of B, as GL is of D. For the same Reason, KF shall be the same Multiple of B, as LH is of D. Therefore because the first EK is the same Multiple of the second B, as the third GL is of the fourth D, and KF, LH are Equimultiples of the second B and sixth D. The first added to the fifth, EF, shall be * the same Multiple of the second B, as the third added to the sixth GH is of the fourth D. *If*, therefore, *the First be the same Multiple of the Second, as the Third is of the Fourth, and there be taken Equimultiples of the First and Third; then will each of the Magnitudes taken be Equimultiples of the Second and Fourth*; which was to be demonstrated.

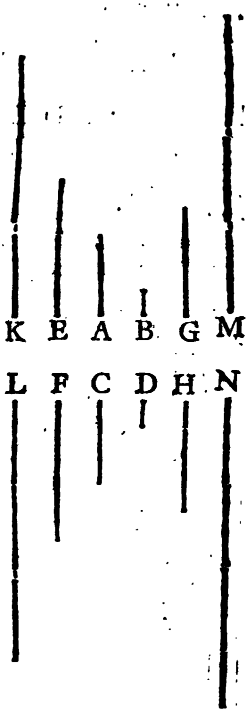
* 2 *of this.*

PROPOSITION IV.

THEOREM.

If the First have the same Proportion to the Second, as the Third to the Fourth; then also shall the Equimultiples of the First and Third have the same Proportion to the Equimultiples of the Second and Fourth, according to any Multiplication whatsoever, if they be so taken as to answer each other.

LET the first A have the same Proportion to the second B as the third C hath to the fourth D; and let E and F, the Equimultiples of A and C, be any how taken; as also G, H, the Equimultiples of B and D. I say E is to G as F is to H.



For take K and L, any Equimultiples of E and F; and also M and N of G and H.

Then because E is the same Multiple of A, as F is of C, and K, L are taken Equimultiples of E, F, K will be * the same Multiple of A, as L is of C. For the same Reason, M is the same Multiple of B as N is of D. And since A is to B, as C is to D, and K and L are Equimultiples of A and C; and also M and N Equimultiples of B and D. If K exceeds M, then † L will exceed N; if equal, equal; or less, less. And K, L are Equimultiples of E, F; and M, N, any other Equimultiples of G H. Therefore, as E is to G, so shall ‡ F be to H. Wherefore, *if the First have the same Proportion to the Second, as the Third to the Fourth;*

* 3 *of this.*

† *Def.* 5 *of this.*

‡ 5 *Def.*

Fourth; then also shall the Equimultiples of the First and Third have the same Proportion to the Equimultiples of the Second and Fourth, according to any Multiplication whatsoever, if they be so taken as to answer each other; which was to be demonstrated.

Because it is demonstrated, if K exceeds M, then L will exceed N; and if it be equal to it, it will be equal; and if less, lesser. It is manifest likewise, if M exceeds K, that N shall exceed L; if equal, equal; but if less, less. And therefore as G is to E, so is * H to F.

* *Def.* 5.

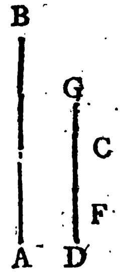
Coroll. From hence it is manifest, if four Magnitudes be proportional, that they will be also inversely proportional.

PROPOSITION. V.

THEOREM.

If one Magnitude be the same Multiple of an other Magnitude, as a Part taken from the one is of a Part taken from the other; then the Residue of the one shall be the same Multiple of the Residue of the other, as the Whole is of the Whole.

LET the Magnitude AB be the same Multiple of the Magnitude CD, as the Part taken away AE is of the Part taken away CF. I say that the Residue EB is the same Multiple of the Residue FD, as the whole AB is of the whole CD.



For let EB be such a Multiple of CG as AE is of CF.

Then because AE is the same Multiple of CF, as EB is of CG, AE will be * the same Multiple of CF, as AB is of GF. But AE and AB are put Equimultiples of CF and CD. Therefore AB is the same Multiple of GF as of CD; and so GF is † equal to CD. Now let CF, which is common, be taken away; and the Residue GC is equal to the Residue DF. And then because AE is the same Multiple of CF, as EB is of CG, and CG is

* 1 *of this.*

† 2 *Axiom of this.*

is equal to DF; AB shall be the same Multiple of CF, as EB is of FD. But AE is put the same Multiple of CF as AB is of CD. Therefore EB is the same Multiple of FD, as AB is of CD: And so the Residue EB is the same Multiple of the Residue FD, as the whole AB is of the whole CD. Wherefore, *if one Magnitude be the same Multiple of another Magnitude, as a Part taken from the one is of a Part taken from the other; then the Residue of the one shall be the same Multiple of the Residue of the other, as the Whole is of the Whole;* which was to be demonstrated.

PROPOSITION VI.

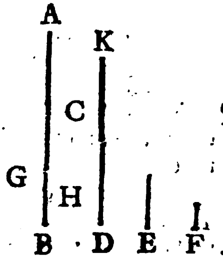
THEOREM.

If two Magnitudes be Equimultiples of two Magnitudes, and some Magnitudes Equimultiples of the same be taken away; then the Residues are either equal to those Magnitudes, or else Equimultiples of them.

LET two Magnitudes AB, CD, be Equimultiples of two Magnitudes E, F, and let the Magnitudes AG, CH, Equimultiples of the same E, F, be taken from AB, CD. I say, the Residues GB, HD, are either equal to E, F, or are Equimultiples of them.

For first, Let GB be equal to E. I say, HD is also equal to F. For let CK be equal to F. Then because AG is the same Multiple of E, as CH is of F; and GB is equal to E; and CK to F; AB will be * the same Multiple of E, as KH is of F. But AB and CD are put Equimultiples of E and F. Therefore KH is the same Multiple of F, as CD is of F.

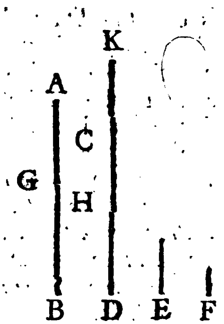
* 1 of this.



And because KH and CD are

Equi-

Equimultiples of F; KH will be equal to CD. Take away CH which is common; then the Residue KC is equal to the Residue HD. But KC is equal to F. Therefore HD is equal to F; and so GB shall be equal to E, and HD to F.



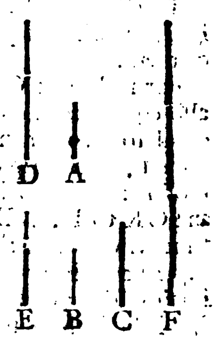
In like Manner we demonstrate, if GB was a Multiple of E, that HD is the like Multiple of F. Therefore, *if two Magnitudes be Equimultiples of two Magnitudes, and some Magnitudes Equimultiples of the same be taken away; then the Residues are either equal to those Magnitudes, or else Equimultiples of them*; which was to be demonstrated.

PROPOSITION VII.

PROBLEM.

Equal Magnitudes have the same Proportion to the same Magnitude; and one and the same Magnitude has the same Proportion to equal Magnitudes.

LET A, B, be equal Magnitudes, and let C be any other Magnitude. I say, A and B have the same Proportion to C; and likewise C has the same Proportion to A as to B.



For take D, E, Equimultiples of A and B, and let F be any other Multiple of C.

Now because D is the same Multiple of A, as E is of B, and *A is equal to B, D shall be also equal to E; but F is a Magnitude taken at Pleasure. Therefore if D exceeds F, then E will exceed F; if D be equal to F, E will be equal to F; and if less, less. But D, E are Equimultiples of A, B; and F is any Multiple of C. Therefore it will be * as A is to C, so is B to C.

* Def. 5.

I say

I say, moreover, that C has the same Proportion to A as to B. For the same Construction remaining, we prove, in like Manner, that D is equal to E. Therefore if F exceeds D, it will also exceed E; if it be equal to D, it will be equal to E; and if it be less than D, it will be less than E. But F is Multiple of C; and D, E, any other Equimultiples of A, B. Therefore as C is to A, so shall * C be to B. Wherefore equal Magnitudes have the same Proportion to the same Magnitude, and the same Magnitude to equal ones; *which was to be demonstrated.* * *Def.* 5.

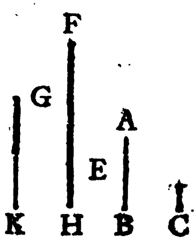
PROPOSITION VIII.

THEOREM.

The greater of any two unequal Magnitudes, has a greater Proportion to some third Magnitude, than the less has; and that third Magnitude hath a greater Proportion to the lesser of the two Magnitudes, than it has to the greater.

LET AB and C be two unequal Magnitudes, whereof AB is the greater; and let D be any third Magnitude. I say, AB has a greater Proportion to D, than C has to D; and D has a greater Proportion to C, than it has to AB.

Because AB is greater than C, make BE equal to C, that is, let AB exceed C by AE; then AE multiplied some Number of Times, will be greater than D. Now let AE be multiply'd until it exceeds D, and let that Multiple of AE, greater than D, be FG. Make GH the same Multiple of EB, and K of C, as FG is of AE. Also, assume L double to D, P triple, and so on, until such a Multiple of D is had, as is the nearest greater than K; let this be N, and let M be a Multiple of D the nearest less than N.



Now because N is the nearest Multiple of D greater

than K, M will not be greater than K, that is, K will not be less than M. And since F G is the same Multiple of A E, as G H is of E B; F G shall be * the same Multiple of A E, as F H is of A B; but F G is the same Multiple of A E, as K is of C; wherefore F H is the same Multiple of A B, as K is of C; that is, F H, K, are Equimultiples of A B and C. Again, Because G H is the same Multiple of E B, as K is of C, and E B is equal to C; G H shall be † equal to K. But K is not less than M. Therefore G H shall not be less than M; but F G is greater than D. Therefore the whole F H will be greater than M and D; but M and D together, are equal to N; because M is a Multiple of D, the nearest lesser than N: Wherefore F H is greater than N. And so since F H exceeds N, and K does not, and F H and K are Equimultiples of A B and C, and N is another Multiple of D; therefore A B will have ‡ a greater Ratio to D, than C has to D. I say, moreover, that D has a greater Ratio to C, than it has to A B; for the same Construction remaining, we demonstrate, as before, that N exceeds K, but not F H. And N is a Multiple of D, and F H, K, are Equimultiples of A B and C. Therefore D has ‡ a greater Proportion to C, than D hath to B. Wherefore *the greater of any two unequal Magnitudes, has a greater Proportion to some third Magnitude, than the less has; and that third Magnitude hath a greater Proportion to the lesser of the two Magnitudes, than it has to the greater.*

* 1 of this.

† 1 Ax.

‡ Def. 7.

PROPOSITION IX.

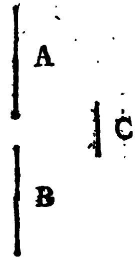
THEOREM

Magnitudes which have the same Proportion to one and the same Magnitude, are equal to one another; and if a Magnitude has the same Proportion to other Magnitudes, these Magnitudes are equal to one another.

LET the Magnitudes A and B have the same Proportion to C. I say, A is equal to B.

For

For if it was not, A and B would not * have the same Proportion to the same Magnitude C; but they have. Therefore A is equal to B. * 8 *of this.*



Again, let C have the same Proportion to A as to B. I say, A is equal to B.

For if it be not, C will not have the same Proportion to A as to B; but it hath: Therefore A is necessarily equal to B. Therefore *Magnitudes that have the same Proportion to one and the same Magnitude, are equal to one another; and if a Magnitude has the same Proportion to other Magnitudes, these Magnitudes are equal to one another*; which was be demonstrated.

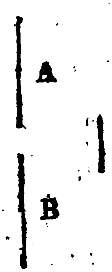
PROPOSITION X.

THEOREM.

Of Magnitudes having Proportion to the same Magnitude, that which has the greater Proportion, is the greater Magnitude: And that Magnitude to which the same bears a greater Proportion, is the lesser Magnitude.

LET A have a greater Proportion to C, than B has to C. I say, A is greater than B.

For if it be not greater, it will either be equal or less. But A is not equal to B, because then both A and B would have * the same Proportion to the same Magnitude C; but they have not. Therefore A is not equal to B: Neither is it less than B; for then A would have † a less Proportion to C than B would have; but it hath not a less Proportion: Therefore A is not less than B. But it has been prov'd likewise not to be equal to it: Therefore A shall be greater than B. * 7 *of this.* † 8 *of this.*



Again, let C have a greater Proportion to B than to A. I say, B is less than A.

For if it be not less, it is greater, or equal. Now
* 6 of this. B is not equal to A; for then C would have * the same Proportion to A as to B; but this it has not. Therefore A is not equal to B; neither is B greater than A; for if it was, C would have a less Proportion to B than to A; but it has not; Therefore B is not greater than A. But it has also been proved not to be equal to it. Wherefore B shall be less than A. Therefore *of Magnitudes having Proportion to the same Magnitude, that which has the greater Proportion, is the greater Magnitude: And that Magnitude to which the same bears a greater Proportion, is the lesser Magnitude*; which was to be demonstrated.

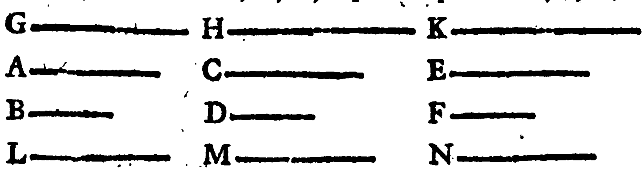
PROPOSITION. XI.

THEOREM.

Proportions that are one and the same to any Third, are also the same to one another.

LET A be to B as C is to D, and C to D as E to F. I say A is to B as E is to F.

For take G, H, K, Equimultiples of A, C, E; and



L, M, N, other Equimultiples of B, D, F. Then because A is to B as C is to D, and there are taken G, H, the Equimultiples of A and C, and L, M any
* 5 Def. of this. Equimultiples of B, D; if G exceeds L, * then H will exceed M; and if G be equal to L, H will be equal to M; and if less, lesser. Again, because as C is to D, so is E to F; and H and K are taken Equimultiples of C and E; as likewise M, N, any Equimultiples of D, F; if H exceeds M *, then K will exceed N; and if H be equal to M, K will be equal to N; and if less, lesser. But if H exceeds M, G will also exceed L; if equal, equal; and if less, less. Wherefore if G exceeds L, K will also exceed N; and if

G

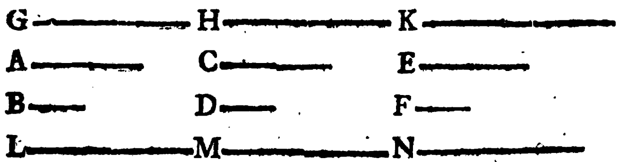
G be equal to L, K will be equal to N; and if less, less. But G, K are Equimultiples of A E, and L, N, any Equimultiples of B, F. Consequently, as A is to B, so * is E to F. Therefore, *Proportions that are one and the same to any Third, are also the same to one another*; which was to be demonstrated. * 5 *Def. of this.*

PROPOSITION XII.

THEOREM.

If any Number of Magnitudes be proportional, as one of the Antecedents is to one of the Consequents, so is all the Antecedents to all the Consequents.

LET there be any Number of proportional Magnitudes, A, B, C, D, E, F; whereof as A is to B,



so C is to D, and so E to F. I say, as A is to B, so are all the Antecedents A, C, E, to all the Consequents B, D, F.

For let G, H, K, be Equimultiples of A, C, E; and L, M, N, any Equimultiples of B, D, F.

Then because as A is to B, so is C to D, and so E to F; and G, H, K, are Equimultiples of A, C, E, and L, M, N, Equimultiples of B, D, F; if G exceeds L, H * will also exceed M, and K will exceed N; if G be equal to L, H will be equal to M, and K to N; and if less, less. Wherefore also, if G exceeds L, then G, H, K, together, will likewise exceed L, M, N, together; and if G be equal to L, then G, H, K, together, will be equal to L, M, N, together; and if less, less: But G, and G, H, K, are Equimultiples of A, and A, C, E; because, if there are any Number of Magnitudes Equimultiples to a like Number of Magnitudes, each to the other, the same Multiple that one Magnitude is of one, so shall † all the Magnitudes be of all. And for the same Reason, L, and L, M, N, * *Def. 5 of this.* † 1 *of this.*

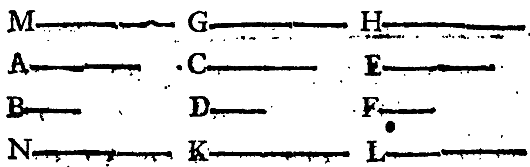
are Equimultiples of B, and B, D, F. Therefore, as
† 5 Def. of this. A is to B, so † is A, C, E, to B, D, F. Wherefore, *if there be any Number of Magnitudes proportional, as one of the Antecedents is to one of the Consequents, so is all the Antecedents to all the Consequents*; which was to be demonstrated.

PROPOSITION XIII.

THEOREM.

If the First has the same Proportion to the Second, as the Third to the Fourth, and if the Third has a greater Proportion to the Fourth than the Fifth to the Sixth; then also shall the First have a greater Proportion to the Second, than the Fifth has to the Sixth.

LET the First A have the same Proportion to the Second B, as the third C has to the Fourth D; and let the Third C have a greater Proportion to the Fourth D, than the Fifth E to the Sixth F. I say,



likewise, that the First A to the Second B has a greater Proportion than the Fifth E to the Sixth F.

For because C has a greater Proportion to D,
* 7. Def. of this. than E has to F; there are * certain Equimultiples of C and E, and others of D and F, such that the Multiple of C may exceed the Multiple of D; but the Multiple of E not that of F. Now let these Equimultiples of C and E, be G and H; and K and L, those of D and F; so that G exceeds K, and H not L; Make M the same Multiple of A, as G is of C; and N the same of B, as K is of D.

Then, because A is to B as C is to D, and M and G are Equimultiples of A, C; and N, K, of B, D:
† 5. Def. If M exceeds N; then † G will exceed K; and if M be equal to N; G will be equal to K; and if less, less. But G does exceed K. Therefore M will also

so

so exceed N. But H does not exceed L. And M, H, are Equimultiples of A, E; and N, L, any others of B, F. Therefore A has * a greater Proportion to B than E has to F. Wherefore, *If the First has the same Proportion to the Second, as the Third to the Fourth; and if the Third has a greater Proportion to the Fourth than the Fifth to the Sixth; then also shall the First have a greater Proportion to the Second, than the Fifth has to the Sixth*; which was to be demonstrated. * 7 Def. of this.

PROPOSITION XIV.

THEOREM.

If the First has the same Proportion to the Second, as the Third has to the Fourth; and if the First be greater than the Third; then will the Second be greater than the Fourth. But if the First be equal to the Third, then the Second shall be equal to the Fourth; and if the first be less than the Third, then the Second will be less than the Fourth.

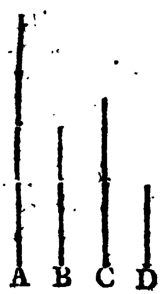
LET the first A have the same Proportion to the second B, as the third C has to the fourth D; And let A be greater than C. I say, B is also greater than D.

For because A is greater than C, and B is any other Magnitude: A will have * a greater Proportion to B than C has to B; but as A is to B, so is C to D; therefore, also, C shall † have a greater Proportion to D than C hath to B. But that Magnitude to which the same bears a greater Proportion, is ‡ the lesser Magnitude: Wherefore D is less than B; and consequently B will be greater than D. In like Manner we demonstrate, if A be equal to C, that B will be equal to D; and if A be less than C, that B will be less than D. Therefore, *if the First has the same Proportion to the Second, as the Third has to the Fourth, and if the First be greater than the Third, then will the Second be greater than the*

* 8 of this.

† 13 of this.

‡ 10 of this.



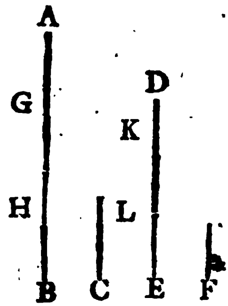
the Fourth. But if the First be equal to the Third, then the Second shall be equal to the Fourth; and if the First be less than the Third, then the Second will be less than the Fourth; which was to be demonstrated.

PROPOSITION XV.

THEOREM.

Parts have the same Proportion as their like Multiples, if taken correspondently.

LET AB be the same Multiple of C, as DE is of F. I say, as C is to F, so is AB to DE.



For because AB and DE are Equimultiples of C and F, there shall be as many Magnitudes equal to C in AB, as there are Magnitudes equal to F in DE. Now, let AB be divided into the Magnitudes AG, GH, HB, each equal to C; and CD into the Magnitudes DK, KL, LE, each equal to F. Then the Number of the Magnitudes AG, GH, HB, will be eqal to the Number of the Magnitudes DK, KL, LE. Now, because AG, GH, HB, are equal, as likewise DK, KL, LE, it shall be * as AG is to DK: So is GH to KL, and so is HB to LE. But as one of the Antecedents is to one of the Consequents, so † all the Antecedents to all the Consequents. Therefore, as AG is to DK, so is AB to DE. But AG is equal to C, and DK to F. Whence, as C is to F, so shall AB be to DE. Therefore, *Parts have the same Proportion as their like Multiples, if taken correspondently*; which was to be demonstrated.

* 7 of this.

† 12 of this.

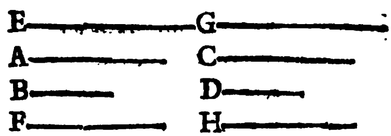
PRO-

PROPOSITION XVI.

THEOREM.

If four Magnitudes of the same Kind are proportional, they also shall be alternately proportional.

LET four Magnitudes ABCD, be proportional; whereof A is to B as C is to D. I say likewise, that they will be alternately proportional, *viz.* as A is to C, so is B to D; for take E, F, Equimultiples of A and B, and G, H, any Equimultiples of C, D.



Then because E is the same Multiple of A, as F is of B, and Parts have the same Proportion * to their like Multiples, if taken correspondently; it shall be as A is to B, so is E to F. But as A is to B, so is C to D. Therefore also as C is to D, so † is E to F. Again, because G, H, are Equimultiples of C and D, and Parts have the same Proportion with their like Multiples, if taken correspondently, it will be as C is to D, so is G to H; but as C is to D, so is E to F. Therefore also as E is to F, so is G to H; and if four Magnitudes be proportional, and the first greater than the third, then the second will be ‡ greater than the fourth; and if the first be equal to the third, the second will be equal to the fourth; and if less, less. Therefore, if E exceeds G, F will exceed H; and if E be equal to G, F will be equal to H; and if less, less. But E, F are any Equimultiples of A, B; and G, H any Equimultiples of C, D. Whence, as A is to C, so shall B be || to D. Therefore, *if four Magnitudes of the same Kind are proportional, they also shall be alternately proportional.*

* 15 *of this.*

† 11 *of this.*

‡ 14 *of this.*

|| *Def.* 5.

PRO-

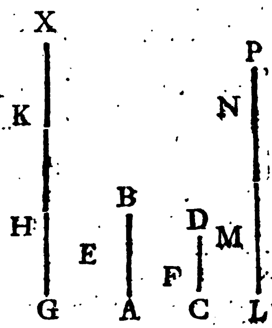
PROPOSITION XVII.

THEOREM.

If Magnitudes compounded, are proportional; they shall also be proportional when divided.

LET the compounded Magnitudes AB, BE, CD, DF, be proportional, that is, let AB be to BE as CD is to DF. I say these Magnitudes divided are proportional, *viz.* as AE is to EB, so is CF to FD.

For let GH, HK, LM, MN, be Equimultiples of AE, EB, CF, FD, and KX, NP any Equimultiples of EB, FD.



Because GH is the same Multiple of AE as HK is of EB; therefore GH * is the same Multiple of AE, as GK is of AB. But GH is the same Multiple of AE, as LM is of CF. Wherefore GK is the same Multiple of AB, as LM is of CF. Again, because LM is the same Multiple of CF as MN is of FD, LM will be * the same Multiple of CF, as LN is of CD. Therefore GK is the same Multiple of AB, as LN is of CD. And so GK, LN, will be Equimultiples of AB, CD. Again, because HK is the same Multiple of EB, as MN is of FD; as likewise KX the same Multiple of EB, as NP is of FD, the compounded Magnitude HX is † also the same Multiple of EB, as MP is of FD. Wherefore, since it is as AB is to BE, so is CD to DF; and GK, LN, are Equimultiples of AB, CD; and also HX, MP, any Equimultiples of EB, FD: If GK exceeds HX, then LN will ‡ exceed MP; and if GK be equal to HX, then LN will be equal to MP; if less, less. Now let GK exceed HX; then if HK, which is common, be taken away, GH shall exceed KX. But when GK exceeds HX, then LN exceeds MP; therefore LN does exceed NP. If MN, which is common,

* 1 *of this.*

† 2 *of this.*

‡ *Def.* 5.

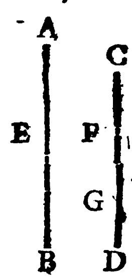
common, be taken away, then LM will exceed NP. And so if GH exceeds KX, then LM will exceed NP. In like manner we demonstrate, if GH be equal to KX, that LM will be equal to NP; and if less, less. But GH, LM, are Equimultiples of AE, CF; and KX, NP, are any Equimultiples of EB, FD. Whence, * as AE is to EB, so CF to FD. Therefore, *if Magnitudes compounded, are proportional; they shall also be proportional when divided*; which was to be demonstrated. * *Def.* 5.

PROPOSITION XVIII.

THEOREM.

If Magnitudes divided be proportional, the same also being compounded, shall be proportional.

LET the divided proportional Magnitudes be AE, EB, CF, FD; that is, as AE is to EB, so is CF to FD. I say they are also proportional when compounded, *viz.* as AB is to BE, so is CD to DF.



For if AB be not to BE, as CD is to DF, AB shall be to BE as CD is to a Magnitude, either greater or less than FD.

First let it be to a lesser, *viz.* to GD. Then because AB is to BE as CD is to DG, compounded Magnitudes are proportional; and consequently * they will be proportional when divided. Therefore AE is to EB as CG is to GD. But (by the *Hyp.*) as AE is to EB, so is CF to FD. Wherefore also as CG is to GD, so † is CF to FD. But the first CG is greater than the third CF; therefore the second DG shall be ‡ greater than the fourth DF. But it is less, which is absurd. Therefore AB is not to BE, as CD is to DG. We demonstrate in the same Manner, that AB to be BE, is not as CD to a greater than DF. Therefore AB to BE, must necessarily be as CD is to DF. And so if Magnitudes divided be proportional, they will also be proportional when compounded; *which was to be demonstrated.*

* 17 *of this.*
† 11 *of this.*
‡ 14 *of this.*

PRO-

PROPOSITION XIX.

THEOREM.

If the Whole be to the Whole, as a Part taken away is to a Part taken away; then shall the Residue be to the Residue, as the Whole is to the Whole.

LET the Whole AB be to the Whole CD, as the Part taken away AE is the Part taken away CF. I say the Residue EB is to the Residue FD, as the Whole AB is to the Whole CD.

For because the Whole AB is to the Whole CD, as AE is to CF; it shall be * alternately as AB is to
* 16 of this. AE, so is CD to CF. Then because compounded Magnitudes, being proportional, will be
† also proportional when divided. As
† 17 of this. BE is to EA, so is DF to FC: And again, it will be by Alternation, as BE to DF, so is EA to FC. But as EA to FC, so (by the *Hyp.*) is AB to CD. And therefore the Residue EB, shall be to the Residue FD, as the Whole AB to the Whole CD. Wherefore, *If the Whole be to the Whole, as a Part taken away is to a Part taken away; then shall the Residue be to the Residue, as the Whole is to the Whole*; which was to be demonstrated.

Coroll. If four Magnitudes be proportional, they will be likewise conversely proportional. For let AB be to BE, as CD to DF; then (by Alternation) it shall be as AB is to CD, so is BE to DF. Wherefore since the Whole AB is to the Whole CD, as the Part taken away BE is to the Part taken away DF; the Residue AE to the Residue CF, shall be as the Whole AB to the Whole CD. And again, (by Inversion and Alternation) as AB is to AE, so is CD to CF, Which is by Converse Ratio.

The Demonstration of Converse Ratio, laid down in this Corollary, is only particular. For Alternation (which is used herein) cannot be apply'd but when the four proportional Magnitudes are all of the same Kind, as will appear from the 4th and 17th Definitions of this Book.

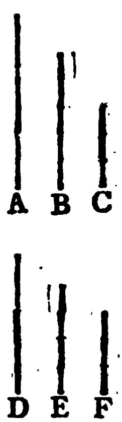
Book. But Converse Ratio may be used when the Terms of the first Ratio are not of the same Kind with the Terms of the latter. Therefore instead of that, it may not be improper to add this Demonstration following: If four Magnitudes are proportional, they will be so conversly: For let AB *be to* BE, *as* CD *to* DF. *And then dividing, it is, as* AE *is to* BE, *so is* CF *to* DF: *And this inversly is, as* BE *is to* AE, *so is* DF *to* CF; *which by compounding becomes, as* AB *is to* AE, *so is* CD *to* CF; *which by the 17th Definition is Converse Ratio. By* S. Cunn.

PROPOSITION XX.

THEOREM.

If there be three Magnitudes, and others equal to them in Number, which being taken two and two in each Order, are in the same Ratio. And if the first Magnitude be greater than the third, then the fourth will be greater than the sixth: But if the first be equal to the third, then the fourth will be equal to the sixth; and if the first be less than the third, the fourth will be less than the sixth.

LET A, B, C, be three Magnitudes, and D, E, F, others equal to them in Number, taken two and two in each Order, are in the same Proportion, *viz.* let A be to B, as D is to E, and B to C, as E to F; and let the first Magnitude A be greater than the third C. I say the fourth D is also greater than the sixth F. And if A be equal to C, D is equal to F. But if A be less than C, D is less than F.



For because A is greater than C, and B is any other Magnitude; and since a greater Magnitude hath * a greater Proportion to to the same Magnitude than a lesser hath, A will have a greater Proportion to B, than C to B. But as A is to B, so is D to E; and inversly, as C is to B, so is F to E. Therefore also D will have a greater Proportion to E, than F has to E. But of Magnitudes having Proportion to the same Magnitude, that which has the greater Proportion

* 8 *of this.*

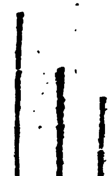
* 10 *of this.* Proportion is * the greater Magnitude. Therefore D is greater than F. In the same Manner we demonstrate, if A be equal to C, then D will be also equal to F; and if A be less than C, then D will be less than F. Therefore, *if there be three Magnitudes, and others equal to them in Number, which being taken two and two in each Order, are in the same Ratio. If the first Magnitude be greater than the third, then the fourth will be greater than the Sixth: But if the first be equal to the third, then the fourth will be equal to the sixth; and if the first be less than the third, the fourth will be less than the sixth*; which was to be demonstrated.

PROPOSITION XXI.

THEOREM.

If there be three Magnitudes, and others equal to them in Number, which taken two and two, are in the same Proportion, and the Proportion be perturbate; if the first Magnitude be greater than the third, then the fourth will be greater than the sixth; but if the first be equal to the third, then is the fourth equal to the sixth; if less, less.

LET three Magnitudes, A, B, C, be proportional; and others D, E, F, equal to them in Number. Let their Analogy likewise be perturbate, *viz.* as A is to B, so is E to F; and as B is to C, so is D to E; if the first Magnitude A be greater than the third C. I say, the fourth D is also greater than the sixth F. And if A be equal to C, then D is equal to F; but if A be less than C, then D is less than F.

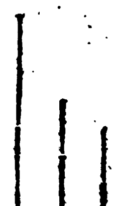


A B C

For since A is greater than C, and B is some other Magnitude, A will have * a greater Proportion to B, than C has to B. But as A is to B, so is E to F; and inversly, as C is to B, so is E to D. Wherefore also E shall have a greater Proportion to F than E to D. But that Magnitude which has a greater Proportion to the same Magnitude, is † the lesser Magnitude. Therefore F is less than D; and so D shall

* 8 *of this.*

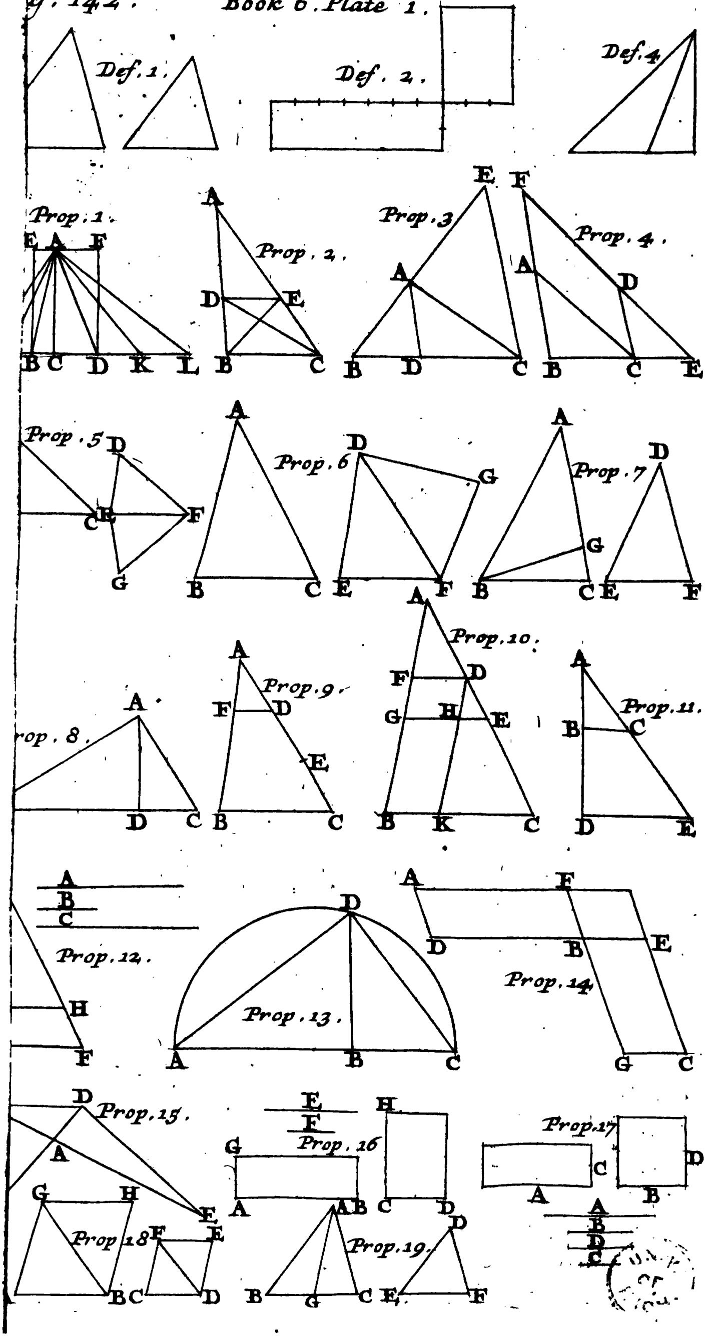
† 10 *of this.*



D E F

be

Book 6. Plate 1.



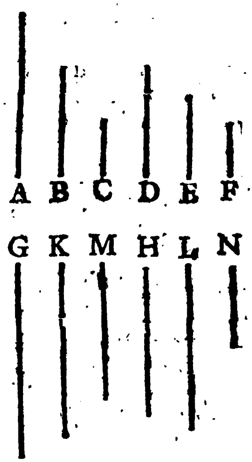
be greater than F. After the same Manner we demonstrate, if A be equal to C, D will be also equal to F; and if A be less than C, D will also be less than F. *If, therefore, there are three Magnitudes, and others equal to them in Number, which taken two and two, are in the same Proportion, and the Proportion be perturbate; if the first Magnitude be greater than the third, then the fourth will be greater than the sixth; but if the first be equal to the third, then is the fourth equal to the sixth; if less, less;* which was to be demonstrated.

PROPOSITION XXII.

THEOREM.

If there be any Number of Magnitudes, and others equal to them in Number, which taken two and two, are in the same Proportion; then they shall be in the same Proportion by Equality.

LET there be any Number of Magnitudes A, B, C, and others D, E, F, equal to them in Number, which taken two and two, are in the same Proportion, that is, as A is to B, so is D to E, and as B is to C, so is E to F. I say, they are also proportional by Equality, viz. as A is to C, so is D to F.



For let G, H, be Equimultiples of A, D; and K, L, any Equimultiples of B, E; and likewise M, N, any Equimultiples of C, F. Then because A is to B, as D is to E, and G, H, are Equimultiples of A, D, and K, L, Equimultiples of B, E; it shall be* as G is to K, so is H to L. For the same Reason also it will be, as K is to M, so is L to N. And since there are three Magnitudes, G, K, M, and others H, L, N, equal to them in Number,

* 4 of this.

ber, which being taken two and two in each Order, are in the same Proportion. If G exceeds M, * H will exceed N; if G be equal to M, then H shall be equal to N; and if G be less than M, H shall be less than N. But G, H, are Equimultiples of A, D, and M, N, any other Equimultiples of C and F. Whence as A is to C, so shall † D be to F. Therefore, *if there be any Number of Magnitudes, and others equal to them in Number, which, taken two and two, are in the same Proportion, then they shall be in the same Proportion by Equality*; which was to be demonstrated.

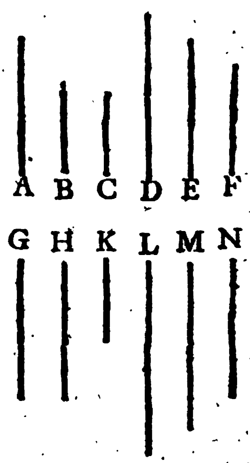
* 20 of this.

† Def. 5 of this.

PROPOSITION XXIII.

THEOREM.

If there be three Magnitudes, and others equal to them in Number, which, taken two and two, are in the same Proportion; and if their Analogy be perturbate, then shall they be also in the same Proportion by Equality.



LET there be three Magnitudes A, B, C, and others equal to them in Number D, E, F, which, taken two and two, are in the same Proportion, and their Analogy be perturbate, that is, as A is to B, so is E to F; and as B is to C, so is D to E. I say, as A is to C, so is D to F.

For let G, H, L, be Equimultiples of A, B, D, and K, M, N, any Equimultiples of C, E, F.

Then because G, H, are Equimultiples of A and B, and since Parts have the same Proportion as their like Multiples when taken correspondently, it shall be * as A is to B, so is G to H; and by the same Reason, as E is to F, so is M to N. But A is to B as E is to F. Therefore, † as G is to H, so is M to N. Again, because B is to C, as D is

* 15 of this.

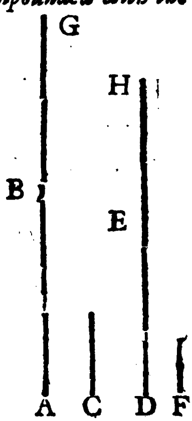
† 11 of this.

is to E, and H, L are Equimultiples of B and D, as likewise K, M any Equimultiples of C, E; it shall be as H to K, so is L to M. But it has been also prov'd, that as G is to H, so is M to N. Therefore, because three Magnitudes, G, H, K, and others, L, M, N, equal to them, in Number, which taken two and two are in the same Proportion, and their Analogy is perturbate; then if G exceeds K, also L * will exceed N; and if G be equal to K, then L will be equal to N; and if G be less than K, L will likewise be less than N. But G, L are Equimultiples of A, D; and K, N Equimultiples of C, F. Therefore, as A is to C, so shall D be to F. Wherefore, *if there be three Magnitudes, and others equal to them in Number, which taken two and two are in the same Proportion; and if their Analogy be perturbate, then shall they be also in the same Proportion by Equality*; which was to be demonstrated. * 21 of this.

PROPOSITION XXIV.

THEOREM.

If the first Magnitude has the same Proportion to the second, as the third to the fourth; and if the fifth has the same Proportion to the second as the sixth has to the fourth, then shall the first, compounded with the fifth, have the same Proportion to the second, as the third compounded with the sixth has to the fourth.



LET the first Magnitude A B have the same Proportion to the second C, as the third D E has to the fourth F. Let also the fifth BG have the same Proportion to the second C, as the sixth E H has to the fourth F. I say, A G the first compounded with the fifth, has the same Proportion to the second C, as D H the third compounded with the sixth, has to the fourth F.

For because BG is to C, as EH is to F, it shall be (inversly) as C is to BG, so is F

L

to EH. Then since AB is to C, as DE is to F, and as C is to BG, so is F to EH; it shall be * by Equality as AB is to BG, so is DE to EH. And because Magnitudes, being divided, are proportional, they shall also be † proportional when compounded. Therefore, as AG is to GB, so is DH to HE: But as GB is ‡ to C, so also is HE to F. Wherefore, by Equality *, it shall be as AG is to C, so is DH to F. Therefore, *if the first Magnitude has the same Proportion to the second, as the third to the fourth; and if the fifth has the same Proportion to the second, as the sixth has to the fourth; then shall the first, compounded with the fifth, have the same Proportion to the second, as the third compounded with the sixth has to the fourth*; which was to be demonstrated.

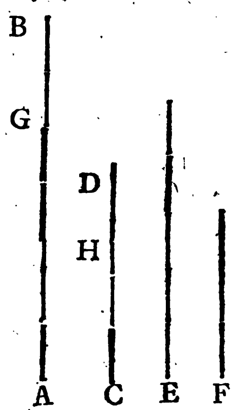
* 22 *of this.*
† 18 *of this.*
‡ *Hyp.*

PROPOSITION XXV.

THEOREM.

If four Magnitudes be proportional, the greatest, and the least of them, will be greater than the other two.

LET four Magnitudes AB, CD, E, F, be proportional, whereof AB is to CD, as E is to F; let AB be the greatest of them, and F the least. I say AB, and F, are greater than CD and E.



For let AG be equal to E, and CH to F. Then because AB is to CD, as E is to F; and since AG, and CH, are each equal to E and F, it shall be as AB is to DC; so is AG to CH. And because the whole AB is to the whole CD, as the Part taken away AG, is to the Part taken away CH; it shall also be * as the Residue GB to the Residue HD; so is the whole AB to the whole CD. But AB is greater than CD; therefore also GB shal be greater than HD. And since AG is equal to E and CH to F, AG and F will be equal to CH and

* 19 *of this.*

and E. But if equal Things are added to unequal Things, the Wholes shall be unequal. Therefore GB, HD being unequal, for GB is the greater. If AG, and F, are added to GB, and CH, and E, to HD; AB and F will necessarily be greater than CD and E. Wherefore, *if four Magnitudes be proportional, the greatest, and the least of them, will be greater than the other two*; which was to be demonstrated.

The End *of the* Fifth Book.

EUCLID's ELEMENTS.

BOOK VI.

DEFINITIONS.

I. S*IMILAR Right-lin'd Figures, are such as have each of their several Angles equal to one another, and the Sides about the equal Angles proportional to each other.*

II. *Figures are said to be reciprocal, when the Antecedent and Consequent Terms of the Ratio's are in each Figure.*

III. *A Right Line is said to be cut into mean and extreme Proportion, when the Whole is to the greater Segment, as the greater Segment is to the lesser.*

IV. *The Altitude of any Figure, is a Perpendicular Line drawn from the Top, or Vertex, to the Base.*

V. *A Ratio is said to be compounded of Ratio's, when the Quantities of the Ratio's being multiplied into one another, do produce a Ratio.*

PRO-

PROPOSITION I.

THEOREM.

Triangles and Parallelograms that have the same Altitude, are to each other as their Bases.

LET the Triangles ABC, ACD, and the Parallelograms E C, C F, have the same Altitude, *viz.* the Perpendicular drawn from the Point A to BD. I say, as the Base BC, is to the Base CD, so is the Triangle A BC, to the Triangle A C D; and so is the Parallelogram E F to the Parallelogram CF.

For produce BD both Ways to the Points H and L, and take GB, GH, any Number of Times equal to the Base BC; and DK, KL, any Number of Times equal to the Base C D, and join AG, AH, AK, AL.

Then because CB, BG, GH, are equal to one another, the Triangles AHG, AGB, ABC, also will be * equal to one another: Therefore the same Multiple that the Base HC is of BC, shall the Triangle AHC be of the Triangle ABC. By the same Reason, the same Multiple that the Base LC is of the Base CD, shall the Triangle ALC be of the Triangle ACD. And if HC be equal to the Base CL, the Triangle AHC is * also equal to the Triangle ALC: And if the Base HC, exceeds the Base CL, then the Triangle AHC, will exceed the Triangle ALC. And if HC be less, then the Triangle AHC, will be less. Therefore since there are four Magnitudes, *viz.* the two Bases BC, CD, and the two Triangles A B C, ACD; and since the Base HC, and the Triangle AHC, are Equimultiplies of the Base BC, and the Triangle ABC: And the Base CL, and the Triangle ALC, are Equimultiplies of the Base CD, and the Triangle ADC. And it has been proved, that if the Base HC, exceeds the Base CL, the Triangle AHC, will exceed the Triangle

* 38. 1.

ALC; and if equal, equal; if less, less. Then as

*Def. 5.5. the Base BC, is to the Base CD, so * is the Triangle ABC, to the Triangle ACD.

†41. 1. And because the Parallelogram EC, is † double to the Triangle ABC; and the Parallelogram FC, double † to the Triangle ACD; and Parts have the same Proportion as their like Multiples. Therefore as the Triangle ABC is to the Triangle ACD, so is the Parallelogram EC to the Parallelogram CF. And so since it has been proved, that the Base BC is to the Base CD, as the Triangle ABC, is to the Triangle ACD; and the Triangle ABC is to the Triangle ACD, as the Parallelogram EC is to the Parallelogram CF; it shall be ‡ as the Base BC, is to

‡11. 5. the Base CD, so is the Parallelogram EC to the Parallelogram FC. Wherefore *Triangles, and Parallelograms, that have the same Altitude, are to each other as their Bases*; which was to be demonstrated.

PROPOSITION II.

THEOREM.

If a Right Line be drawn parallel to one of the Sides of a Triangle, it shall cut the Sides of the Triangle proportionally; and if the Sides of the Triangle be cut proportionally, then a Right Line joining the Points of Section, shall be parallel to the other Side of the Triangle.

LET DE be drawn parallel to BC, a Side of the Triangle ABC. I say, DB is to DA, as CE is to EA.

For let BE, CD, be joined.

*37. 1. Then the Triangle BDE is * equal to the Triangle CDE, for they stand upon the same Base DE, and are between the same Parallels DE and BC; and ADE is some other Triangle. But equal Magni-

†7. 5. tudes have † the same Proportion to one and the same Magnitude. Therefore as the Triangle BDE is to the Triangle ADE, so is the Triangle CDE to the Triangle ADE.

But as the Triangle BDE, is to the Triangle

‡1 of this. ADE, so ‡ is BD to DA; for since they have the

same

same Altitude, *viz.* a Perpendicular drawn from the Point E to A B, they are to each other as their Bases. And for the same Reason, as the Triangle C D E, is to the Triangle A D E, so is C E to E A: And therefore as B D is to D A, so * is C E to E A. * 11. 5.

And if the Sides A B, A C, of the Triangle A B C, be cut proportionally, that is, so that B D be to D A, as C E is to C A; and if D E be joined, I say, D E is parallel to B C.

For the same Construction remaining, because B D is to D A, as C E is to E A; and B D is † to D A, as the Triangle B D E is to the Triangle A D E; and C E is to E A, as the Triangle C D E is to the Triangle A D E: It shall be as the Triangle B D E, is to the Triangle A D E, so is * the Triangle C D E to the Triangle A D E. And since the Triangles A D E, C D E, have the same Proportion to the Triangle A D E, the Triangle B D E, shall be † equal to the Triangle C D E; and they have the same Base D E: But equal Triangles being upon the same Base, are ‡ between the same Parallels; therefore D E is parallel to B C. Wherefore *if a Right Line be drawn parallel to one of the Sides of a Triangle, it shall cut the Sides of the Triangle proportionally; and if the Sides of the Triangle be cut proportionally, then a Right Line joining the Points of Section, shall be parallel to the other Side of the Triangle*; which was to be demonstrated.

† 1 of this. † 9. 5. ‡ 39. 1.

PROPOSITION III.

THEOREM.

If one Angle of a Triangle be bisected, and the Right Line that bisects the Angle, cuts the Base also; then the Segments of the Base will have the same Proportion, as the other Sides of the Triangle. And if the Segments of the Base have the same Proportion that the other Sides of the Triangle have; then a Right Line drawn from the Vertex, to the Point of Section of the Base, will bisect the Angle of the Triangle.

LET there be a Triangle A B C, and let its Angle A B C, be * bisected by the Right Line A D. I say, as B D is to D C, so is B A to A C. * 9. 1.

* 31. 1. For thro' C draw * CE parallel to DA, and produce BA till it meets CE in the Point E.

Then because the Right Line AC, falls on the
† 29. 1. Parallels AD, EC, the Angle ACE, will be † equal to the Angle CAD: But the Angle CAD (by the Hypothesis) is equal to the Angle BAD. Therefore the Angle BAD, will be equal to the Angle ACE. Again, because the Right Line BAE, falls on the Parallels AD, EC, the outward Angle BAD, is † equal to the inward Angle AEC; but the Angle ACE, has been proved equal to the Angle BAD: Therefore ACE, shall be equal to AEC; and so the
‡ 6. 1. Side AE is equal ‡ to the Side AC. And because the Line AD is drawn parallel to CE, the Side of the
* 2 *of this.* Triangle BCE, it shall be * as BD is to DC, so is BA to AE; but AE is equal to AC. Therefore as
† 7. 5. BD is to DC, so is † BA to AC.

And if BD be to DC, as BA is to AC; and the Right Line AD be joined, then, I say, the Angle BAC, is bisected by the Right Line AD.

For the same Construction remaining, because BD is to DC, as BA is to AC; and as BD is to DC, so
‡ 2 *of this.* is ‡ BA to AE; for AD is drawn parallel to one Side EC of the Triangle BCE, it shall be as BA is to AC, so is BA to AE. Therefore AC is equal to AE; and accordingly the Angle AEC, is equal to the Angle ECA: But the Angle AEC, is equal
* 29. 1. * to the outward Angle BAD; and the Angle ACE, equal * to the alternate Angle CAD. Wherefore the Angle BAD is also equal to the Angle CAD; and so the Angle BAC is bisected by the Right Line AD. Therefore *if the Angle of a Triangle be bisected, and the Right Line that bisects the Angle, cuts the Base also; then the Segments of the Base will have the same Proportion as the other Sides of the Triangle. And if the Segments of the Base have the same Proportion that the other Sides of the Triangle have; then a Right Line drawn from the Vertex, to the Point of Section of the Base, will bisect the Angle of the Triangle*; which was to be demonstrated.

PRO-

PROPOSITION IV.

THEOREM.

The Sides about the equal Angles of equiangular Triangles, are proportional; and the Sides which are subtended under the equal Angles, are homologous, or of like Ratio.

LET ABC, DCE, be equiangular Triangles, having the Angle ABC equal to the Angle DCE; the Angle ACB equal to the Angle DEC, and the Angle BAC equal to the Angle CDE. I say, the Sides that are about the equal Angles of the Triangles ABC, DCE, are proportional; and the Sides that are subtended under the equal Angles, are homologous, or of like Ratio.

Set the Side BC, in the same Right Line with the Side CE; and because the Angles ABC, ACB, are * less than two Right Angles, and the Angle ACB is equal to the Angle DEC, the Angles ABC, DEC, are less than two Right Angles. And so BA, ED, produced, will meet † each other; let them be produced, and meet in the Point F. Then because the Angle DCE, is equal to the Angle ABC, BF shall be ‡ parallel to DC. Again, because the Angle ACB is equal to the Angle DEC, the Side AC will be ‡ parallel to the Side FE; therefore FACD is a Parallelogram; and consequently FA is * equal to CE, and AC to FD; and because AC is drawn parallel to FE, the Side of the Triangle FBE, it shall be † as BA is to AF, so is BC to CE; and (by Alternation) as BA is to BC, so is CD to CE. Again; because CD is parallel to BF, it shall be † as BC is to CE, so is FD to DE; but FD is equal to AC. Therefore as BC is to CE, so is ‡ AC to DE: And so (by Alternation) as BC is to CA, so is CE to ED. Wherefore because it is demonstrated that AB is to BC, as DC is to CE; and as BC is to CA, so is CE to ED; it shall be * by Equality, as BA is to AC, so is CD to DE. Therefore *the Sides about the equal Angles of equiangular Triangles, are proportional; and the*

* 17. 1.
† 13 Ax.
‡ 28. 1.
* 34. 1.
† 2 of this.
‡ 7. 5.
* 21. 1.

the Sides, which are subtended under the equal Angles, are homologous, or of like Ratio; which was to be demonstrated.

PROPOSITION V.

THEOREM.

If the Sides of two Triangles are proportional, the Triangles shall be equiangular; and their Angles, under which the homologous Sides are subtended, are equal.

LET there be two Triangles, ABC, DEF, having their Sides proportional, that is, let AB be to BC, as DE is to EF; and as BC to CA, so is EF to FD. And also as BA to CA, so ED to DF. I say, the Triangle ABC is equiangular to the Triangle DEF; and the Angles equal, under which the homologous Sides are subtended, *viz.* the Angle ABC, equal to the Angle DEF; and the Angle BCA equal to the Angle EFD, and the Angle BAC equal to the Angle EDF.

For at the Points E and F, with the Line EF, make * the Angle FEG, equal to the Angle ABC; and the Angle EFG, equal to the Angle BCA: Then the remaining Angle BAC, is † equal to the remaining Angle EGF.

* 23. 1.

† *Cor.* 32. 1.

And so the Triangle ABC is equiangular to the Triangle EGF; and consequently, the Sides that are subtended under the equal Angles, are proportional. Therefore as AB is to BC, so is ‡ GE to EF; but as AB is to BC, so is DE to EF: Therefore as DE is to EF, so is * GE to EF. And since DE, EG, have the same Proportion to EF, DE shall be † equal to EG. For the same Reason, DF is equal to FG; but EF is common. Then because the two Sides DE, EF, are equal to the two Sides GE, EF, and the Base DF is equal to the Base FG, the Angle DEF is ‡ equal to the Angle GEF; and the Triangle DEF equal to the Triangle GEF; and the other Angles of the one, equal to the other Angles of the other, which are subtended by the equal Sides. Therefore the Angle DEF is equal to the Angle GEF, and the Angle EDF equal to the Angle EGF;

‡ 4 *of this.*

* 11. 5.

† 9. 5.

‡ 8. 1.

EGF; and because the Angle DEF is equal to the Angle GEF; and the Angle GEF equal to the Angle ABC; therefore the Angle ABC shall be also equal to the Angle FED: For the same Reason, the Angle ACB shall be equal to the Angle DFE; as also the Angle A equal to the Angle D; therefore the Triangle ABC will be equiangular to the Triangle DEF. Wherefore *if the Sides of two Triangles are proportional, the Triangles shall be equiangular; and their Angles, under which the homologus Sides are subtended, are equal*; which was to be demonstrated.

PROPOSITION VI.

THOREM.

If two Triangles have one Angle of the one equal to one Angle of the other; and if the Sides about the equal Angles be proportional, then the Triangles are equiangular, and have those Angles equal, under which are subtended the homologous Sides.

LET there be two Triangles ABC, DEF, having one Angle BAC of the one equal to the Angle EDF of the other; and let the Sides about the equal Angles be proportional, *viz.* let AB be to AC, as ED is to DF. I say, the Triangle ABC is equiangular to the Triangle DEF; and the Angle ABC equal to the Angle DEF; and the Angle ACB equal to the Angle DFE.

For at the Points D and F, with the Right Line DF, make * the Angle FDG equal to either of the Angles BAC, EDF; and the Angle DFG equal to the Angle ACB, * 23. 1.

Then the other Angle B, is † equal to the other Angle G; and so the Triangle ABC, is equiangular to the Triangle DGF; and consequently, as BA is to AC, so is ‡ GD to DF: But (by the *Hyp.*) as BA is to AC, so is ED to DF. Therefore as ED is * to DF, so is GD to DF; whence ED is † equal to DG, and DF is common; therefore the two Sides ED, DF, are equal to the two Sides GD, DF; and the Angle EDF, equal to the Angle GDF: † Cor. 32. 1. ‡ 4 of this. * 11. 5. † 9. 5.

Con-

* 4. 1. Consequently the Base EF is * equal to the Base FG, and the Triangle DEF equal to the Triangle GDF; and the other Angles of the one, equal to the other Angles of the other, each to each; under which the equal Sides are subtended. Therefore the Angle DFG is equal to the Angle DFE, and the Angle G, equal to the Angle E; but the Angle DFG, is equal to the Angle ACB: Wherefore the Angle ACB is equal to the Angle DFE; and the Angle
† *by Hyp.* BAC is † also equal to the Angle EDF: Therefore the other Angle at B is equal to the other Angle at E; and so the Triangle ABC is equiangular to the Triangle DEF. Therefore *if two Triangles have one Angle of the one, equal to one Angle of the other; and if the Sides about the equal Angles be proportional, then the Triangles are equiangular; and have those Angles equal, under which are subtended the homologous Sides;* which was to be demonstrated.

PROPOSITION VII.

THEOREM.

If there are two Triangles, having one Angle of the one equal to one Angle of the other, and the Sides about other Angles proportional; and if the remaining third Angles are either both less, or both not less than Right Angles, then shall the Triangles be equiangular and have those Angles equal, about which are the proportional Sides.

LET two Triangles ABC, DEF, have one Angle of the one, equal to one Angle of the other, *viz.* the Angle BAC equal to the Angle EDF; and let the Sides about the other Angles ABC, DEF, be proportional; *viz.* as DE is to EF, so let AB be to BC; and let the other Angles at C and F, be both less, or both not less than Right Angles. I say, the Triangle ABC is equiangular to the Triangle DEF; and the Angle ABC is equal to the Angle DEF; as also the other Angle at C, equal to the other Angle at F.

For if the Angle ABC be not equal to the Angle DEF, one of them will be the greater, which let be

ABC

ABC. Then at the Point B, with the Right Line AB, make * the Angle ABG equal to the Angle DEF. * 23. 1.

Now because the Angle A is equal to the Angle D, and the Angle ABG, equal to the Angle DEF; the remaining Angle AGB, is † equal to the remaining Angle DFE: And therefore the Triangle ABG, is equiangular to the Triangle DEF; and so as AB is to BG, so is ‡ DE to EF; but as DE is to EF, so is * AB to BC. Therefore as AB is to BC, so is AB to BG; and since AB has the same Proportion to BC, that it has to BG, BC shall be † equal to BG; and consequently the Angle at C equal to the Angle BGC. Wherefore each of the Angles BCG, or BGC is less than a Right Angle; and consequently, AGB is greater than a Right Angle. But the Angle AGB has been proved equal to the Angle at F; therefore the Angle at F, is greater than a Right Angle: But (by the *Hyp.*) it is not greater, since C is not greater than a Right Angle, which is absurd. Wherefore the Angle ABC is not unequal to the Angle DEF; and so it must be equal to the same; but the Angle at A is equal to that at D; wherefore the Angle remaining at C is equal to the remaining Angle at F; and consequently the Triangle ABC is equiangular to the Triangle DEF. Therefore *if there are two Triangles having one Angle of the one, equal to one Angle of the other, and the Sides about other Angles proportional; and if the remaining third Angles are either both less, or both not less than Right Angles, then shall the Triangles be equiangular; and have those Angles equal, about which are the proportional Sides*; which was to be demonstrated.

† Cor. 32. 1.
‡ 4 of this.
* by Hyp.
† 9. 5.

PRO-

PROPOSITION VIII.

THEOREM.

If a Perpendicular be drawn, in a Right-angl'd Triangle, from the Right Angle to the Base, then the Triangles on each Side of the Perpendicular are similar both to the Whole, and also to one another.

LET ABC be a Right-angl'd Triangle, whose Right Angle is BAC; and let the Perpendicular AD be drawn from the Point A to the Base BC. I say, the Triangles ABD, ADC, are similar to one another, and to the whole Triangle ABC.

For because the Angle BAC is equal to the Angle ADB, for each of them is a Right Angle, and the Angle at B is common to the two Triangles ABC, ABD, the remaining Angle ACB shall be * equal to the remaining Angle BAD. Therefore the Triangle ABC is equiangular to the Triangle ABD; and so as † BC, which subtends the Right Angle of the Triangle ABC, is to BA, subtending the Right Angle of the Triangle ABD, so is AB subtending the Angle C of the Triangle ABC to DB, subtending an Angle equal to the Angle C, *viz.* the Angle BAD, of the Triangle ABD. And so moreover is AC to AD, subtending the Angle B, which is common to the two Triangles. Therefore the Triangle ABC is ‡ equiangular to the Triangle ABD; and the Sides about the equal Angles are proportional. Wherefore the Triangle ABC is ‡ similar to the Triangle ABD. By the same Way we demonstrate, that the Triangle ADC is also similar to the Triangle ABC. Wherefore each of the Triangles ABD, ADC, is similar to the whole Triangle.

* *Cor.* 32. 1.
† 4 *of this.*
‡ *Def.* 1 *of this.*

I say, the said Triangles are also similar to one another.

For because the Right Angle BDA is equal to the Right Angle ADC, and the Angle BAD has been prov'd equal to the Angle C; it follows, that the remaining Angle at B shall be equal to the remaining Angle DAC. And so the Triangle ABD is equiangular to the Triangle ADC. Wherefore as † BD subtending

† 4 *of this.*

subtending the Angle BAD of the Triangle ABD is to DA, subtending the Angle at C of the Triangle ADC, which is equal to the Angle BAD, so is AD subtending the Angle B of the Triangle ABD to DC, subtending the Angle DAC equal to the Angle B. And moreover, so is BA to AC, subtending the Right Angles at D; and consequently the Triangle ABD is similar to the Triangle ADC. Wherefore, *if a Perpendicular be drawn, in a Right-angl'd Triangle, from the Right Angle to the Base, then the Triangles on each Side of the Perpendicular are similar both to the Whole, and also to one another*; which was to be demonstrated.

Coroll. From hence it is manifest, that the Perpendicular drawn in a Right-angl'd Triangle from the Right Angle to the Base, is a Mean proportional between the Segments of the Base. Moreover, either of the Sides containing the Right Angle is a Mean proportional between the whole Base, and that Segment thereof which is next to the Side.

PROPOSITION. IX.

PROBLEM.

To cut off any Part requir'd from a given Right Line.

LET AB be a Right Line given; from which must be cut off any requir'd Part; suppose a third.

Draw any Right Line AC from the Point A, making an Angle at Pleasure with the Line AB. Assume any Part D in the Line AC, make * DE, EC, each equal to AD, join BC, and draw † DF thro' D, parallel to BC. * 3. 1. † 31. 1.

Then because FD is drawn parallel to the Side BC of the Triangle ABC, it shall be ‡ as CD is to DA, so is BF to FA. But CD is double to DA. Therefore BF shall be double to FA; and so BA is triple to AF. Wherefore there is cut off AF, a third Part requir'd of the given Right Line AB; *which was to be done.* ‡ 2 *of this.*

PRO-

PROPOSITION X.

PROBLEM.

To divide a given undivided Right Line, as another given Right Line is divided.

LET AB be a given undivided Right Line, and AC a divided Line. It is requir'd to divide AB, as AC is divided.

Let AC be divided in the Points D and E, and so placed, as to contain any Angle with AB. Join the Points C and B; thro' D and E let DF, EG, be
* 31. 1. drawn * parallel to BC; and thro' D, draw DHK, parallel to AB.

Then FH, HB, are each of them Parallelograms;
† 34. 1. and so DH is † equal to FG, and HK to GB. And because HE is drawn parallel to the Side KC, of the
‡ 2 of this. Triangle DKC, it shall be ‡ as CE is to ED; so is KH to HD. But KH is equal to BG, and HD to GF. Therefore, as CE is to ED, so is BG to GF. Again, because FD is drawn parallel to the Side EG, of the Triangle AGE, as ED is to DA, so shall ‡ GF be to FA. But it has been prov'd, that CE is to ED as BG is to GF. Therefore, as CE is to ED, so is BG to GF; and as ED is to DA, so is GF to FA. Wherefore the given undivided Line AB, is divided as the given Line AC is; *which was to be done.*

PROPOSITION XI.

PROBLEM.

Two Right Lines being given, to find a third proportional to them.

LET AB, AC, be two given Right Lines, so placed, as to make any Angle with each other. It is requir'd to find a third proportional to AB, AC.

Produce AB, AC, to the Points D and E; make BD equal to AC, join the Points B, C, and
* 31. 1. draw * the Right Line DE thro' D parallel to BC.

Then

Then because BC is drawn parallel to the Side DE, of the Triangle ADE, it shall be * as AB is to BD, so is AC to CE. But BD is equal to AC. Hence as AB is to AC, so is AC to CE. Therefore a third proportional CE is found to two given Right Lines AB, AC; *which was to be done.* * 2 of this.

PROPOSITION XII.

PROBLEM.

Three Right Lines being given, to find a fourth proportional to them.

LET A, B, C, be three Right Lines given. It is requir'd to find a fourth proportional to them.

Let DE and DF be two Right Lines, making any Angle EDF with each other. Now make DG equal to A, GE equal to B, DH equal to C, and draw the Line GH, as also * EF through E, parallel to GH. * 31. 1.

Then because GH is drawn parallel to EF, the Side of the Triangle DEF, it shall be as DG is to GE, so is DH to HF. But DG is equal to A, GE to B, and DH to C. Consequently as A is to B, so is C to HF. Therefore the Right Line HF, a fourth Proportional to the three given Right Lines A, B, C, is found; *which was to be done.*

PROPOSITION XIII.

PROBLEM.

To find a Mean proportional between two given Right Lines.

LET the two given Right Lines be AB, BC. It is requir'd to find a Mean proportional between them. Place AB, BC, in a direct Line, and on the whole AC describe the Semicircle ADC, and draw * BD at Right Angles to AC from the Point B, and let AD, DC, be joined. * 11. 1.

Then because the Angle ADC, in a Semicircle, is † a Right Angle, and since the Perpendicular DB is drawn from the Right Angle to the Base; therefore † 31. 3.

* Cor. 8 of this. therefore DB is * a Mean Proportional between the Segments of the Base AB, BC. Wherefore a Mean proportional between the two given Lines AB, BC, is found; *which was to be done.*

PROPOSITION XIV.

THEOREM.

Equal Parallelograms having one Angle of the one equal to one Angle of the other, have the Sides about the equal Angles reciprocal; and those Parallelograms that have one Angle of the one equal to one Angle of the other, and the Sides that are about the equal Angles reciprocal, are equal between themselves.

LET AB, BC, be equal Parallelograms, having the Angles at B equal; and let the Sides DB, BE, be in one strait Line; then also will * the Sides FB, BC, be in one strait Line. I say, the Sides of the Parallelograms AB, BC, that are about the equal Angles, are reciprocal; that is, as DB is to BE, so is GB to BF.

* 14. 1.

For let the Parallelogram FE be compleated.

Then because the Parallelogram AB is equal to the Parallelogram BC, and FE is some other Parallelogram; it shall be as AB is to FE, so is † BC to FE; but as AB is to FE, so is ‡ DB to BE; and as BC is to FE, so is GB to BF. Therefore, as DB is to BE, so is GB to BF. Wherefore the Sides of the Parallelograms AB, BC, that are about the equal Angles, are reciprocally proportional.

† 7. 5.
‡ 1 of this.

And if the Sides that are about the equal Angles are reciprocally proportional, *viz.* if BD be to BE as GB is to BF. I say the Prallelogram AB is equal to the Parallelogram BC.

For since DB is to BE as GB is to BF, and DB to BE as the Parallelogram AB ‡ to the Parallelogram FE, and GB ‡ to BF as the Parallelogram BC to the Parallelogram FE; it shall be as AB is to FE, so is BC to FE. Therefore the Parallelogram AB is equal to the Parallelogram BC. And so *equal Parallelograms having one Angle of the one equal to one Angle of the other, have the Sides about the equal Angles*

gles reciprocal; and those Parallelograms that have one Angle of the one equal to one Angle of the other, and the Sides that are about the equal Angles reciprocal, are equal between themselves; which was to be demonstrated.

PROPOSITION XV.

THEOREM.

Equal Triangles having one Angle of the one equal to one Angle of the other, have their Sides about the equal Angles reciprocal; and those Triangles that have one Angle of the one equal to one Angle of the other, and have also the Sides about the equal Angles reciprocal, are equal between themselves.

LET the equal Triangles ABC, ADE, have one Angle of the one equal to one Angle of the other, *viz.* the Angle BAC equal to the Angle DAE. I say the Sides about the equal Angles are reciprocal, that is, as CA is to AD, so is EA to AB.

For place CA and AD in one strait Line, then EA and AB shall be * also in one strait Line, and let BD be joined. Then because the Triangle ABC is equal to the Triangle ADE, and ABD is some other Triangle, the Triangle CAB shall be † to the Triangle BAD, as the Triangle ADE is to the Triangle BAD. But as the Triangle CAB is to the Triangle BAD, so is CA ‡ to AD; and as the Triangle EAD is to the Triangle BAD, so is ‡ EA to AB. Therefore as CA is to AD, so is EA to AB. Wherefore the Sides of the Triangles ABC, ADE, about the equal Angles, are reciprocal.

* 14. 1.

† 7. 5.

‡ 1 *of this.*

And if the Sides about the equal Angles of the Triangles ABC, ADE, be reciprocal, *viz.* if CA be to AD as EA is to AB. I say the Triangle ABC is equal to the Triangle ADE.

For, again let BD be joined. Then because CA is to AD as EA is to AB, and CA to AD ‡ as the Triangle ABC to the Triangle BAD, and EA to AB ‡ as the Triangle EAD to the Triangle BAD; therefore, as the Triangle ABC is to the Triangle BAD, so shall the Triangle EAD be to the Triangle

BAD. Whence the Triangles ABC, ADE, have the same Proportion to the Triangle BAD: And so the Triangle ABC is equal to the Triangle ADE. Therefore *equal Triangles having one Angle of the one equal to one Angle of the other, have their Sides about the equal Angles reciprocal; and those Triangles that have one Angle of the one equal to one Angle of the other, and have also the Sides about the equal Angles reciprocal, are equal between themselves*; which was to be demonstrated.

PROPOSITION XVI.

THEOREM.

If four Right Lines be proportional, the Rectangle contained under the Extremes is equal to the Rectangle contained under the Means; and if the Rectangle contained under the Extremes be equal to the Rectangle contained under the Means, then are the four Right Lines proportional.

LET four Right Lines AB, CD, E, F, be proportional, so that AB be to CD, as E is to F. I say the Rectangle contain'd under the Right Lines AB and F, is equal to the Rectangle contain'd under the Right Lines CD and E.

For draw AG, CH, from the Points A, C, at Right Angles to AB and CD, and make AG equal to F, and CH equal to E, and let the Parallelograms BG, DH, be compleated.

Then because AB is to CD as E is to F, and since CH is equal to E, and AG to F, it shall be as AB is to CD, so is CH to AG. Therefore the Sides that are about the equal Angles of the Parallelograms BG, DH are reciprocal; and since those Parallelograms are equal *, that have the Sides about the equal Angles reciprocal. Therefore the Parallelogram BG is equal to the Parallelogram DH. But the Parallelogram BG is equal to that contain'd under AB and F; for AG is equal to F, and the Parallelogram DH equal to that contain'd under CD and E, since CH is equal to E. Therefore the Rectangle contain'd under AB and F is equal to that contain'd under CD and E.

* 14 *of this.*

And

And if the Rectangle contained under AB and F, be equal to the Rectangle contained under CD and E, I say the four Right Lines are Proportionals, *viz.* as AB is to CD, so is E to F.

For, the same Construction remaining, the Rectangle contained under AB and F is equal to that contained under CD and E; but the Rectangle contained under AB and F is the Rectangle BG; for AG is equal to F: And the Rectangle contained under CD and E is the Rectangle DH, for CH is equal to E. Therefore the Parallelogram BG, shall be equal to the Parallelogram DH, and they are equiangular; but the Sides of equal and equiangular Parallelograms, which are about the equal Angles, are * reciprocal. Wherefore as AB is to CD, so is CH to AG; but CH is equal to E, and AG to F; therefore as AB is to CD, so is E to F. Wherefore, *if four Right Lines be proportional, the Rectangle contained under the Extremes, is equal to the Rectangle contained under the Means; and if the Rectangle contained under the Extremes be equal to the Rectangle contained under the Means, then are the four Right Lines proportional*; which was to be demonstrated. * 14 *of this.*

PROPOSITION XVII.

THEOREM.

If three Right Lines be proportional, the Rectangle contained under the Extremes, is equal to the Square of the Mean; and if the Rectangle under the Extremes, be equal to the Square of the Mean, then the three Right Lines are proportional.

LET there be three Right Lines A, B, C, proportional; and let A be to B, as B is to C. I say, the Rectangle contained under A and C, is equal to the Square of B.

For make D equal to B.

Then because A is to B as B is to C, and B is equal to D, it shall be * as A is to B, so is D to C. * 7. 5.
But if four Right Lines be Proportionals, the Rectangle contained under the Extremes is † equal to the Rectangle under the Means. Therefore the Rectangle † 16 *of this.*

M 3 gle

gle contained under A and C, is equal to the Rectangle under B and D; but the Rectangle under B and D is equal to the Square of D, for B is equal to D. Wherefore the Rectangle contained under A, C, is equal to the Square of B.

And if the Rectangle contained under A, C, be equal to the Square of B; I say, as A is to B, so is B to C.

For the same Construction remaining, the Rectangle contained under A and C is equal to the Square of B; but the Square of B is the Rectangle contained under B, D, for B is equal to D, and the Rectangle contained under A, C, shall be equal to the Rectangle contained under B, D. But if the Rectangle contained under the Extremes, be equal to the Rectangle contained under the Means, the four Right Lines shall be † Proportionals. Therefore A is to B as D is to C; but B is equal to D. Wherefore A is to B, as B is to C. Therefore *if three Right Lines be proportional, the Rectangle contained under the Extremes, is equal to the Square of the Mean; and if the Rectangle under the Extremes, be equal to the Square of the Mean, then the three Right Lines are proportional*; which was to be demonstrated.

† 16 of this.

PROPOSITION XVIII.

PROBLEM.

Upon a given Right Line, to describe a Right-lin'd Figure similar, and similarly situate to a Right-lin'd Figure given.

LET AB be the Right Line given, and CE the Right-lin'd Figure. It is required to describe upon the Right Line AB a Figure similar, and similarly situate to the Right Line Figure CE.

* 23. 1. Join DF, and make * at the Points A and B, with the Line AB, the Angles GAB, ABG, each equal to the Angles C and CDE. Whence the other Angle CFD is † equal to the other Angle AGB; and so the Triangle FCD is equiangular to the Triangle GAB; and consequently, as FD is to GB, so is ‡ FC to GA; and so is CD to AB. Again, make the

† Cor. 32. 1.

‡ 4 of this.

the Angles BGH, GBH, at the Points B and G, with the Right Line BG, each equal to the Angles EFD, EDF; then the remaining Angle at E, is † equal to the remaining Angle at H. Therefore the Triangle FDE, is equiangular to the Triangle GBH; and consequently, as FD is to GB, so is ‡ FE to GH; and so ED to HB. But it has been proved that FD is to GB, as FC is to GA, and as CD to AB. And therefore as FC is to AG, so is * CD to AB; and so FE to GH; and so ED to HB. And because the Angle CFD is equal to the Angle AGB; and the Angle DFE equal to the Angle BGH; the whole Angle CFE shall be equal to the whole Angle AGH. By the same Reason, the Angle CDE is equal to the Angle ABH; and the Angle at C equal to the Angle A; and the Angle E equal to the Angle H. Therefore the Figure AH is equiangular to the Figure CE; and they have the Sides about the equal Angles proportional. Consequently, the Right-lin'd Figure AH will be † similar to the Right-lin'd Figure CE. Therefore there is described upon the given Right Line AB, the Right-lin'd Figure AH similar, and similarly situate to the given Right-lin'd Figure CE; *which was to be done.*

† Cor. 32. 1.

‡ 4 of this.

* 11. 5.

† Def. 1 of this.

PROPOSITION XIX.

THEOREM.

Similar Triangles are in the duplicate Proportion of their homologous Sides.

LET ABC, DEF, be similar Triangles, having the Angle B equal to the Angle E; and let AB be to BC as DE is to EF, so that BC be the Side homologous to EF. I say, the Triangle ABC, to the Triangle DEF, has a duplicate Proportion to that of the Side BC to the Side EF.

For take * BG a third Proportional to BC and EF; that is, let BC be to EF, as EF is to BG, and join GA.

* 11 of this.

Then because AB is to BC, as DE is to EF; it shall be (by Alternation) as AB is to DE, so is BC to EF; but as BC is to EF, so is EF to BG. There-

 fore

† 11. 5. fore as AB is to DE, so is † EF to BG; consequently, the Sides that are about the equal Angles of the Triangles ABG, DEF, are reciprocal: But those Triangles that have one Angle of the one, equal to one Angle of the other; and the Sides about the equal Angles reciprocal, are ‡ equal. Therefore the Triangle ABG, is equal to the Triangle DEF; and because BC is to EF, as EF is to BG, and if three Right Lines be proportional, the first has * a duplicate Proportion to the third, of what it has to the second. BC to BG shall have a duplicate Proportion of that which BC has to EF; but the Triangle ABG is equal to the Triangle DEF. Therefore the Triangle ABC, to the Triangle DEF, shall be in the duplicate Proportion of that which the Side BC has to the Side EF. Wherefore *similar Triangles are in the duplicate Proportion of their homologous Sides*; which was to be demonstrated.

‡ 15 of this.

* Def. 10. 5.

Coroll. From hence it is manifest, if three Right Lines be proportional, then as the first is to the third, so is a Triangle made upon the first to a similar, and similarly described Triangle upon the second: Because it has been proved, as CB is to BG, so is the Triangle ABC to the Triangle ABG, that is, to the Triangle DEF; *which was to be demonstrated.*

PROPOSITION XX.

THEOREM.

Similar Polygons are divided into similar Triangles, equal in Number, and homologous to the Wholes; and Polygon to Polygon, is in the duplicate Proportion of that which one homologous Side has to the other.

LET ABCDE, FGHKL, be similar Polygons; and let the Side AB be homologous to the Side FG. I say, the Polygons ABCDE, FGHKL, are divided into equal Numbers of similar Triangles, and homologous to the Wholes; and the Polygon ABCDE, to the Polygon FGHKL, is in the duplicate Proportion of that which the Side AB has to the Side FG.

For let BE, EC, GL, LH, be joined. Then

Then because the Polygon ABCDE is similar to the Polygon FGHKL, the Angle BAE is equal to the Angle GFL; and BA is to AE as GF is to FL. Now since ABE, FGL, are two Triangles, having one Angle of the one equal to one Angle of the other, and the Sides about the equal Angles proportional; the Triangle ABE will be * equiangular to the Triangle FGL; and so also similar to it. Therefore the Angle ABE, is equal to the Angle FGL; but the whole Angle ABC is † equal to the whole Angle FGH, because of the Similiarity of the Polygons. Therefore the remaining Angle EBC is equal to the remaining Angle LGH: And since (by the Similarity of the Triangles ABE, FGL) as EB is to EA, so is LG to GF; And since also (by the Similarity of the Polygons) AB is to BC, as FG is to GH; it shall be ‡ by Equality of Proportion, as EB is to BC, so is LG to GH, that is, the Sides about the equal Angles EBC, LGH, are proportional. Wherefore the Triangle EBC is equiangular to the Triangle LGH; and consequently also similar to it. For the same Reason, the Triangle ECD, is likewise similar to the Triangle LHK; therefore the similar Polygons ABCDE, FGHKL, are divided into equal Numbers of similar Triangles.

* 6 of this.

† Def. 1 of this.

‡ 21. 5.

I say, they are also homologous to the Wholes, that is, that the Triangles are proportional; and the Antecedents are ABE, EBC, ECD, and their Consequents FGL, LGH, LHK. And the Polygon ABCDE, to the Polygon FGHKL, is in the duplicate Proportion of an homologous Side of the one, to an homologous Side of the other, that is, AB to FG.

For because the Triangle ABE is similar to the Triangle FGL, the Triangle ABE, shall be * to the Triangle FGL, in the duplicate Proportion of BE to GL: For the same Reason, the Triangle BEC, to the Triangle GLH, is * in a duplicate Proportion of BE to GL; Therefore the Triangle ABE is † to the Triangle FGL, as the Triangle BEC is to the Triangle GLH. Again, because the Triangle EBC is similar to the Triangle LGH; the Triangle EBC to the Triangle LGH, shall be in the duplicate Proportion of the Right Line CE to the Right Line HL; and so likewise the Triangle ECD to the Triangle

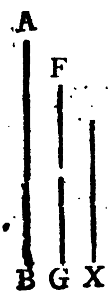
* 19 of this.

† 11. 5.

angle LHK, shall be in the duplicate Proportion of CE to HL. Therefore the Triangle BEC is to the Triangle LGH, as the Triangle CED is to the Triangle LHK. But it has been proved, that the Triangle EBC is to the Triangle LGH, as the Triangle ABE is to the Triangle FGL: Therefore as the Triangle ABE is to the Triangle FGL, so is the Triangle BEC to the Triangle GHL; and so is the Triangle ECD to the Triangle LHK. But as one of the Antecedents is to one of the Consequents, so are ‡ all the Antecedents to all the Consequents. Wherefore as the Triangle ABE is to the Triangle FGL, so is the Polygon ABCDE to the Polygon FGHKL: But the Triangle ABE to the Triangle FGL, is in the duplicate Proportion of the homologous Side AB to the homologous Side FG; for similar Triangles are in the duplicate Proportion of the homologous Sides. Wherefore the Polygon ABCDE, to the Polygon FGHKL, is in the duplicate Proportion of the homologous Side AB to the homologous Side FG. Therefore *similar Polygons are divided into similar Triangles, equal in Number, and homologous to the Wholes; and Polygon to Polygon, is in the duplicate Proportion of that which one homologous Side has to the other*; which was to be demonstrated.

‡ 12. 5.

It may be demonstrated after the same Manner that similar quadrilateral Figures are to each other in the duplicate Proportion of their homologous Sides; and this has been already proved in Triangles.



Coroll. 1. Therefore universally similar Right-lin'd Figures, are to one another in the duplicate Proportion of their homologous Sides; and if X be taken a third Proportional to AB and FG, then AB will have to X a duplicate Proportion of that which AB has to FG; and a Polygon to a Polygon, and a quadrilateral Figure to a quadrilateral Figure, will be in the duplicate Proportion of that which one homologous Side has to the other; that is, AB to FG; but this has been proved in Triangles.

2. Therefore universally it is manifest, if three Right-Lines be proportional, as the first is to the third, so is a Figure described upon the first, to a similar and similarly

similarly described Figure on the second; *which was to be demonstrated.*

PROPOSITION XXI.

THEOREM.

Figures that are similar to the same Right-lin'd Figure, are also similar to one another.

LET each of the Right-lin'd Figures A, B, be similar to the Right-lin'd Figure C. I say, the Right-lin'd Figure A, is also similar to the Right-lin'd Figure B.

For because the Right-lin'd Figure A is similar to the Right-lin'd Figure C, it shall be * equiangular thereto; and the Sides about the equal Angles proportional. Again, because the Right-lin'd Figure B is similar to the Right-lin'd Figure C, it shall * be equiangular thereto; and the Sides about the equal Angles will be proportional. Therefore each of the Right-lin'd Figures A, B, are equiangular to C, and they have the Sides about the equal Angles proportional. Wherefore the Right-lin'd Figure A is equiangular to the Right-lin'd Figure B; and the Sides about the equal Angles are proportional; wherefore A is similar to B; *which was to be demonstrated.* * *Def.* 1. *of this.*

PROPOSITION XXII.

THEOREM.

If four Right Lines be proportional, the Right-lin'd Figures similar and similarly described upon them, shall be proportional; and if the similar Right-lin'd Figures similarly described upon the Lines, be proportional, then the Right Lines shall also be proportional.

LET four Right Lines AB, CD, EF, GH, be proportional; and as AB, is to CD, so let EF be to GH.

Now let the similar Figures KAB, LCD, be similarly described * upon AB, CD; and the similar Figures MF, NH, similarly described upon the Right Lines * 18 *of this.*

Lines E F, G H. I say, as the Right-lin'd Figure K A B is to the Right-lin'd Figure L C D, so is the Right-lin'd Figure M F to the Right-lin'd Figure N H.

* 11 of this. For take * X a third Proportional to AB, C D, and O a third Proportional to EF, GH.

Then because AB is to CD, as EF is to GH; and
† 22. 5. as CD is to X, so is GH to O; it shall be † by Equality of Proportion, as AB is to X, so is E F to O. But AB is to X, as the Right-lin'd Figure K A B is
‡ Cor. 20. of this. ‡ to the Right-lin'd Figure L C D; and as E F is to O, so is ‡ the Right-lin'd Figure M F, to the Right-lin'd Figure N H. Therefore as the Right-lin'd Figure K A B is to the Right-lin'd Figure L C D, so is
* 11. 5. * the Right-lin'd Figure M F to the Right-lin'd Figure N H.

And if the Right-lin'd Figure K A B be to the Right-lin'd Figure L C D, as the Right-lin'd Figure M F is to the Right-lin'd Figure N H; I say, as A B is to C D, so is E F to G H.

† 12 of this. For make † E F to P R, as A B is to C D, and describe upon P R a Right-lin'd Figure S R similar, and alike situate, to either of the Figures M F and N H.

Then because A B is to C D, as E F is to P R, and there are described upon A B, C D, similar and alike situate Right-lin'd Figures K A B, L C D, and upon E F, P R, similar and alike situate Figures M F, S R; it shall be (by what has been already proved) as the Right-lin'd Figure K A B is to the Right-lin'd Figure L C D, so is the Right-lin'd Figure M F to the Right-lin'd Figure R S; But (by the *Hyp.*) as the Right-lin'd Figure K A B is to the Right-lin'd Figure L C D, so is the Right-lin'd Figure M F to the Right-lin'd Figure N H. Therefore as the Right-lin'd Figure M F is to the Right-lin'd Figure N H, so is the Right-lin'd Figure M F to the Right-lin'd Figure S R: And since the Right-lin'd Figure M F has the same Proportion to N H, as it hath to S R, the Right-lin'd Figure N H
‡ 9. 5. shall be ‡ equal to the Right-lin'd Figure S R; it is also similar to it, and alike described; therefore G H is equal to P R. And because A B is to C D, as E F is to P R; and P R is equal to G H, it shall be as A B is to C D, so is E F to G H. Therefore, *if four Right Lines be proportional, the Right-lin'd Figures, similar and similarly described upon them, shall be proportional;*

and

and if the similar Right-lin'd Figures similarly described upon the Lines, be proportional, then the Right Lines shall also be proportional; which was to be demonstrated.

LEMMA.

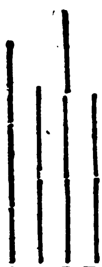
Any three Right Lines A, B, *and* C, *being given, the Ratio of the first* A *to the third* C, *is equal to the Ratio compounded of the Ratio of the first* A *to the second* B, *and of the Ratio of the second* B *to the third* C.

FOR Example, let the Number 3 *be the Exponent, or Denominator of the Ratio of* A *to* B; *that is, let* A *be three times* B, *and let the Number* 4 *be the Exponent of the Ratio of* B *to* C; *then the Number* 12 *produced by the Multiplication of* 4 *and* 3, *is the compounded Exponent of the Ratio of* A *to* C: *For since* A *contains* B *thrice, and* B *contains* C *four times,* A *will contain* C *thrice four times; that is,* 12 *times. This is also true of other Multiples, or Submultiples; but this Theorem may be universally demonstrated thus: The Quantity of the Ratio of* A *to* B, *is the Number* $\frac{A}{B}$ *viz. which multiplying the Consequent, produces the Antecedent. So likewise the Quantity of the Ratio of* B *to* C, *is* $\frac{B}{C}$. *And these two Quantities multiplied by each other, produce the Number* $\frac{A \times B}{B \times C}$, *which is the Quantity of the Ratio that the Rectangle comprehended under the Right Lines* A *and* B, *has to the Rectangle comprehended under the Right Lines* B *and* C; *and so the said Ratio of the Rectangle under* A *and* B, *to the Rectangle under* B *and* C, *is that which in the Sense of* Def. 5 *of this Book, is compounded of the Ratio's of* A *to* B, *and* B *to* C; *but (by* 1. 6.*) the Rectangle contained under* A *and* B, *is to the Rectangle contained under* B *and* C, *as* A *is to* C; *therefore the Ratio of* A *to* C, *is equal to the*

Ratio

Ratio compounded of the Ratio's of A *to* B, *and of* B *to* C.

If any four Right Lines A, B, C, *and* D, *be proposed, the Ratio of the first* A *to the fourth* D *is equal to the Ratio compounded of the Ratio of the first* A *to the second* B, *and of the Ratio of the second* B *to the third* C, *and of the Ratio of the third* C *to the fourth* D.



A B C D

For in three Right Lines A, C, *and* D, *the Ratio of* A *to* D, *is equal to the Ratio's compounded of the Ratio's of* A *to* C, *and of* C *to* D; *and it has been already demonstrated, that the Ratio of* A *to* C *is equal to the Ratio compounded of the Ratio's of* A *to* B, *and* B *to* C. *Therefore the Ratio of* A *to* D *is equal to the Ratio compounded of the Ratio's of* A *to* B, *of* B *to* C, *and of* C *to* D. *After the same Manner we demonstrate, in any Number of Right Lines, that the Ratio of the first to the last is equal to the Ratio compounded of the Ratio's of the first to the second, of the second to the third, of the third to the fourth, and so on to the last.*

This is true of any other Quantities besides Right Lines, which will be manifest, if the same Number of Right Lines A, B, C, &c. *as there are Magnitudes be assumed in the same Ratio,* viz. *so that the Right Line* A *is to the Right Line* B, *as the first Magnitude is to the second, and the Right Line* B *to the Right Line* C, *as the second Magnitude is to the third, and so on. It is manifest (by* 22. 5.*) by Equality of Proportion, that the first Right Line* A *is to the last Right Line, as the first Magnitude is to the last; but the Ratio of the Right Line* A *to the last Right Line, is equal to the Ratio compounded of the Ratio's of* A *to* B, B *to* C, *and so on to the last Right Line: But (by the* Hyp.*) the Ratio of any one of the Right Lines to that nearest to it, is the same as the Ratio of a Magnitude of the same Order to that nearest it. And therefore the Ratio of the first Magnitude to the last, is equal to the Ratio compounded of the Ratio's of the first Magnitude to the second, of the second to the third, and so on to the last;* which was to be demonstrated.

PRO-

PROPOSITION XXIII.

THEOREM.

Equiangular Parallelograms have the Proportion to one another that is compounded of their Sides.

LET AC, CF, be equiangular Parallelograms, having the Angle BCD equal to the Angle ECG. I say, the Parallelogram AC, to the Parallelogram CF, is in the Proportion compounded of their Sides, *viz.* compounded of the Proportion of BG to CG, and of DC to CE.

For let BC be placed in the same Right Line with CG.

Then DC shall be * in a strait Line with CE, and compleat the Parallelogram DG; and then † as BC is to CG, so is some Right Line K to L; and as DC is to CE, so let L be to M. * 14. 1. † 12 *of this.*

Then the Proportions of K to L, and of L to M, are the same as the Proportions of the Sides, *viz.* of BC to CG, and DC to CE; but the Proportion of K to M is ‡ compounded of the Proportion of K to L, and of the Proportion of L to M. Wherefore also K to M hath a Proportion compounded of the Sides. Then because BC is to CG as the Parallelogram AC is * to the Parallelogram CH: And since BC is to CG as K is to L, it shall be † as K is to L, so is the Parallelogram AC, to the Parallelogram CH. Again, because DC is to CE as the Parallelogram CH is to the Parallelogram CF; and since as DC is to CE, so is L to M. Therefore as L is to M, so shall † the Parallelogram CH be to the Parallelogram CF; and consequently since it has been proved that K is to L, as the Parallelogram AC is to the Parallelogram CH, and as L is to M, so is the Parallelogram CH to the Parallelogram CF; it shall be ‡ by Equality of Proportion, as K is to M, so is the Parallelogram AC to the Parallelogram CF; but K to M hath a Proportion compounded of the Sides: Therefore also the Parallelogram AC, to the Parallelogram CF, hath a Proportion compounded of the Sides. Wherefore *equiangular Parallelograms have* ‡ *Com. proceed.* * 1 *of this.* † 11. 5. ‡ 20. 5.

the

the Proportion to one another that is compounded of their Sides; which was to be demonstrated.

PROPOSITION XXIV.

THEOREM.

In every Parallelogram, the Parallelograms that are about the Diameter, are similar to the whole, and also to one another.

LET ABCD be a Parallelogram, whose Diameter is AC; and EG, HK, be Parallelograms about the Diameter AC. I say the Parallelograms EG, HK, are similar to the Whole ABCD, and also to each other.

For because EF is drawn parallel to BC, the Side
* 2 *of this.* of the Triangle ABC, it shall be * as BE to EA, so is CF to FA. Again, because FG is drawn parallel to CD, the Side of the Triangle ACD, it shall be as CF to FA, so is * DG to GA. But CF is to FA, (as has been prov'd) as BE is to EA. There-
† 11. 5. fore, as BE is to EA, so is † DG to GA; and by
‡ 18. 5. compounding, as BA is to AE, so is ‡ DA to AG; and by Alternation, as BA is to AD, so is EA to AG. Therefore the Sides of the Parallelograms ABCD, EG, which are about the common Angle BAD, are proportional. And because GF is paral-
* 29. 1. lel to DC, the Angle AGF is * equal to the Angle ADC, and the Angle GFA equal to the Angle DCA; and the Angle DAC is common to the two Triangles ADC, AGF. Wherefore the Triangle ADC will be equiangular to the Triangle AGF. For the same Reason, the Triangle ACB is equiangular to the Triangle AFE. Therefore the whole Parallelogram ABCD is equiangular to the Parallelogram EG;
† 4 *of this.* and so AD is to DC as AG is † to GF. But DC is to CA as GF is to FA; and AC is to CB, as AF is to FE; and moreover, CB is to BA as FE is to EA. Wherefore, since it has been prov'd, that DC is to CA, as GF is to FA, and AC is to CB, as AF is to FE; it shall be, by Equality of Proportion, as DC is to CB, so is GF to FE. Therefore the Sides that are about the equal Angles of the Parallelograms

AB

ABCD, EG, are proportional; and accordingly the Parallelogram ABCD is similar to the Parallelogram EG. For the same Reason the Parallelogram ABCD is similar to the Parallelogram KH. Therefore both the Parallelograms EG, HK, are similar to the Parallelogram ABCD. But Right-lin'd Figures that are similar to the same Right-lin'd Figure, are * similar to one another. Therefore the Parallelogram EG is similar to the Parallelogram HK. And so *in every Parallelogram, the Parallelograms that are about the the Diameter are similar to the Whole, and also to one another*; which was to be demonstrated. * 21 *of this.*

PROPOSITION XXV.

PROBLEM.

To describe a Right-lin'd Figure similar to a Right-lin'd Figure which shall be given, and equal to another Right-lin'd Figure given.

LET ABC be a given Right-lin'd Figure, to which it is requir'd to describe another similar and equal to D.

On the Side BC of the given Figure ABC, make * the Parallelogram BE equal to the Right-lin'd Figure ABC; and on the Side CE make * the Parallelogram CM equal to the Right-lin'd Figure D, in the Angle FCE, equal to the Angle CBL. Then BC, CF, as also LE, EM, will be † in two strait Lines. Find ‡ GH a Mean proportional between BC, CF, and on GH let there be describ'd * the Right-lin'd Figure KGH similar and alike situate to the Right-lin'd Figure ABC. * 44. 1. † 14. 1. ‡ 13 *of this.* * 18 *of this.*

And then because BC is to GH, as GH is to CF, and since when three Right Lines are proportional, the first is to the third as the Figure describ'd on the first is † to a similar and alike situate Figure describ'd on the second, it shall be as BC is to CF, so is the Right-lin'd Figure ABC to the Right-lin'd Figure KGH. But as BC is to CF, so is ‡ the Parallelogram BE to the Parallelogram EF. Therefore, as the Right-lin'd Figure ABC is to the Right-lin'd Figure KGH, so is the Parallelogram BE to the Parallelogram EF. † *Cor.* 20 *of this.* ‡ 1 *of this.*

N Where-

Wherefore, (by Alternation) as the Right-lin'd Figure ABC is to the Parallelogram BE, so is the Right-lin'd Figure KGH to the Parallelogram EF. But the Right-lin'd Figure ABC is equal to the Parallelogram BE. Therefore the Right-lin'd Figure KGH is also equal to the Parallelogram EF. But the Parallelogram EF is equal to the Right-lin'd Figure D. Therefore the Right-lin'd Figure KGH is equal to D. But KGH is similar to ABC. Consequently there is describ'd the Right-lin'd Figure KGH similar to the given Figure ABC, and equal to the given Figure D; *which was to be done.*

PROPOSITION XXVI.

THEOREM.

If from a Parallelogram be taken away another similar to the Whole, and in like manner situate, having also an Angle common with it, then is that Parallelogram about the same Diameter with the Whole.

LET the Parallelogram AF be taken away from the Parallelogram ABCD similar to ABCD, and in like manner situate, having the Angle DAB common. I say the Parallelogram ABCD is about the same Diameter with the Parallelogram AF.

For if it be not, let AHC be the Diameter of the Parallelogram BD, and let GF be produc'd to H; also let HK be drawn parallel to AD, or BC.

Then because the Parallelogram ABCD is about the same Diameter as the Parallelogram KG, the Parallelogram ABCD shall be * similar to the Parallelogram KG; and so as DA is to AB, so is † GA to AK. But because of the Similarity of the Parallelograms ABCD, EG, as DA is to AB, so is GA to AE. And therefore as GA is ‡ to AE, so is GA to AK. And since GA has the same Proportion to AK as to AE, AE is ‖ equal to AK, the less to a greater, which is absurd. Therefore the Parallelogram ABCD is not about the same Diameter as the Parallelogram AH. And therefore it will be about the same Diameter with the Parallelogram AF. Therefore, *if from a Parallelogram be taken away another similar*

* 24 of this.
† 1 Def. of this.
‡ 11. 5.
‖ 9. 5.

to the Whole, and in like manner situate, having also an Angle common with it, then is that Parallelogram abo the same Diameter with the Whole; which was to demonstrated.

PROPOSITION XXVII.

THEOREM.

Of all Parallelograms apply'd to the same Right Line, and wanting in Figure by Parallelograms similar and alike situate, describ'd on the half Line, the greatest is that which is apply'd to the half Line, being similar to the Defect.

LET AB be a Right Line, bisected in the Point C, and let the Parallelogram AD be apply'd to the Right Line AB, wanting in Figure the Parallelogram CE, similar and alike situate to that describ'd on half of the Right Line AB. I say, AD is the greatest of all Parallelograms apply'd to the Right Line AB, wanting in Figure by Parallelograms similar and alike situate to CE. For let the Parallelogram AF be apply'd to the Right Line AB, wanting in Figure the Parallelogram HK, similar and alike situate to the Parallelogram CE. I say the Parallelogram AD is greater than the Parallelogram AF.

For because the Parallelogram CE is similar to the Parallelogram HK, they stand * about the same Diameter, let DB their Diameter be drawn, and the Figure describ'd. Then since the Parallelogram CF is † equal to FE, let HK, which is common, be added; and the Whole CH is equal to the Whole KE. But CH is ‡ equal to CG, because the Right Line AC is equal to CB. Therefore the whole AF is equal to the Gnomon LNM; and so CE, that is, the Parallelogram AD is greater than the Parallelogram AF. Therefore, *of all Parallelograms apply'd to the same Right Line, and wanting in Figure by Parallelograms similar and alike situate, describ'd on the half Line, the greatest is that which is apply'd to the half Line, being similar to the Defect*; which was to be demonstrated.

* 26 *of this.*

† 43. 1.

‡ 36. 1.

PROPOSITION XXVIII.

PROBLEM.

To a Right Line given to apply a Parallelogram equal to a Right Line Figure given, deficient by a Parallelogram, which is similar to another given Parallelogram; but it is necessary that the Right-lin'd Figure given, to which the Parallelogram to be apply'd must be equal, be not greater than the Parallelogram which is apply'd to the half Line, since the Defects must be similar, viz. *the Defect of the Parallelogram apply'd to the half Line, and the Defect of the Parallelogram to be apply'd.*

LET AB be a given Right Line, and let the given Right-lin'd Figure, to which the Parallelogram to be apply'd to the Right Line AB must be equal, be C, which must not be greater than the Parallelogram apply'd to the half Line, the Defects being similar; and let D be the Parallelogram, to which the Defect of the Parallelogram to be apply'd is similar. Now it is requir'd to apply a Parallelogram equal to the given Right-lin'd Figure C to the given Right Line AB, deficient by a Parallelogram similar to D.

* 18 *of this.* Let AB be bisected in E, and on EB describe * the Parallelogram EBFG, similar and alike situate to D, and compleat the Parallelogram AG.

Now AG is either equal to C, or greater than it, because of the Determination. If AG be equal to C, what was propos'd will be done; for the Parallelogram AG is apply'd to the Right Line AB, equal to the given Right-lin'd Figure C, deficient by the Parallelogram EF, similar to the Parallelogram D. But if it be not equal, then HE is greater than C; but EF is equal to HE. Therefore EF shall also be

† 25 *of this.* greater than C. Now make † the Parallelogram KLMN similar and alike situate to D, and equal to the Excess, by which EF exceeds C. But D is similar to EF. Wherefore KM shall also be similar to EF. Therefore let the Right Line KL be homologous to GE, and LM to GF. Then because EF is equal to C and KM together, EF will be greater than KM; and

and so the Right Line GE is greater than KL, and GF than LM. Make GX equal to KL, and GO equal to LM, and compleat the Parallelogram XGOP. Therefore XO is equal and similar to KM, but KM is similar to EF; therefore XO is * similar to EF, and so XO is † about the same Diameter with FE: Let GPB be their Diameter, and the Figure be described. * 21 *of this.* † 26 *of this.*

Then since EF is equal to C and KM together, and XO is equal to KM, the Gnomen ΥΦΨ remaining, is equal to the remaining Figure C; and because OR is equal to XS, let SR, which is common, be added; then the Whole OB is equal to the Whole XB; but XB is equal to TE, since the Side AE is equal to the Side EB. Wherefore TE is equal to OB. Add XS, which is common, and then the whole TS is equal to the whole Gnomen ΥΦΨ; but the Gnomen ΥΦΨ has been proved equal to C; and so TS shall be equal to C; and so the Parallelogram TS is apply'd to the Right Line AB, equal to the given Right-lin'd Figure C, and deficient by a Parallelogram SR, similar to the Parallelogram D, because SR is similar to FE; *which was to be done.*

PROPOSITION XXIX.

THEOREM.

To a Right Line given, to apply a Parallelogram equal to a Right-lin'd Figure given, exceeding by a Parallelogram, which shall be similar to another given Parallelogram.

LET AB be a given Right Line, and let C be the given Right-lin'd Figure to which that to be apply'd to AB must be equal. Likewise let D be the Parallelogram to which the exceeding Parallelogram is to be similar; it is required to apply a Parallelogram to the Right Line AB, equal to the given Right-lined Figure C, exceeding by a Parallelogram similar to D.

Bisect AB in E, and let the Parallelogram EL be described * upon the Right Line EB, similar and alike situate to D; and that † the Parallelogram GH equal * 18. 1. † 25 *of this.*

to E L and C together, but similar to D, and alike situate. Therefore G H is similar to E L; let K H be a Side homologous to F L, and K G to F E. Then because the Parallelogram G H is greater than the Parallelogram E L, the Right Line K H will be greater than F L, and K G greater than F E. Let F L, F E, be produced, and let F L M be equal to K H, F E N equal to K G, and compleat the Parallelogram M N. Therefore M N is equal and similar to G H; but G H

‡ 21 of this. is similar to E L, and so M N shall be ‡ similar to

* 26 of this. E L; and accordingly E L is * about the same Diameter with M N. Let F X be their Diameter, and describe the Figure.

Then since G H is equal to E L and C together, as likewise to M N; therefore M N shall be equal to E L and C. Let E L, which is common, be taken away, then the Gnomon ΥΦΨ remaining, is equal to C; and since A E is equal to E B, the Parallelogram A N will be also equal to the Parallelogram E P, that is, to L O; and if E X, which is common, be added, then the whole Parallelogram A X is equal to the Gnomon ΥΦΨ but the Gnomon ΥΦΨ is equal to C. Therefore A X shall be also equal to C. Wherefore the Parallelogram A X is apply'd to the given Right Line A B, equal to the given Right-lin'd Figure C, and exceeding by the Parallelogram P O, similar to the Parallelogram D; *which was to be done.*

PROPOSITION. XXX.

PROBLEM.

To cut a given terminate Right Line according to extreme and mean Ratio.

LET A B a given terminate Line; it is required to cut the same according to extreme and mean Ratio.

* 46. 1. Describe * B C the Square of A B, and apply the Parallelogram C D to A C, equal to the Square B C,

† 29 of this. exceeding † by the Figure A D similar to B C; but B C is a Square, therefore A D shall also be a Square.

Now because B C is equal to C D, take away C E, which is common; then B F remaining shall be equal

to

to AD remaining; but BF is equiangular to AD; therefore the Sides that are about the equal Angles are ‡ reciprocally proportional; and so as FE is to ED, so is AE to EB, but FE is * equal to AC, that is, to AB, and ED to AE. Wherefore as BA is to AE, so is AE to EB, but AB is greater than AE; therefore AE is † greater than EB; and so the Right Line AB is cut according to extreme and mean Ratio in the Point E; and AE is the greater Segment thereof; *which was to be done.* ‡ 14 *of this.* * 34. 1. † 14. 5.

Otherwise thus: Let AB be the Right Line given; it is required to cut the same into extreme and mean Ratio.

Divide ‡ AB so in C, that the Rectangle contained under AB, BC, be equal to the Square of AC. ‡ 11. 2.

Then because the Rectangle under AB, BC, is equal to the Square of AC, it shall be * as BA is to AC, so is AC to CB; and so the Right Line AB is cut into mean and extreme Ratio; *which was to be done.* * 17 *of this.*

PROPOSITION. XXXI.

THEOREM.

Any Figure described upon the Side of a Right-angled Triangle subtending the Right Angle, is equal to the Figures described upon the Sides containing the Right Angle, being similar and alike situate to the former Figure.

LET ABC be a rectangular Triangle, having the Right Angle BAC. I say the Figure described on BC, is equal to the two Figures together described on BA, AC, which are similar and alike situate to the Figure described on BC.

For draw the Perpendicular AD.

Then because the Right Line AD is drawn in the Right-angled Triangle ACB, from the Right Angle A, perpendicular to the Base BC; the Triangles ABD, ADC, which are about the Perpendicular AD, will be * similar to the whole Triangle ABC, and also to each other. Then because the Triangle ABC is similar to the Triangle ABD, it shall be * as CB is to BA, so * 8 *of this.*

is BA to BD; and ſince when three Right Lines are proportional, the firſt ſhall be † to the third, as a Figure deſcribed on the firſt, to a ſimilar and alike ſituate Figure deſcribed on the ſecond. Wherefore as CB is to BD, ſo is a Figure deſcribed on CB to a ſimilar and alike ſituate Figure deſcribed on BA. For the ſame Reaſon as BC is to CD, ſo is a Figure deſcribed on BC to one deſcribed on CA. Wherefore alſo, as BC is to BD and DC together, ſo is ‡ the Figure deſcribed on BC, to thoſe two together that are deſcribed ſimilar and alike ſituate on BA, AC; but BC is equal to BD and DC together: Therefore the Figure deſcribed on BC is equal to thoſe together deſcribed on BA, AC, ſimilar and alike ſituate to that on BC. Wherefore, *any Figure deſcribed upon the Side of a Right-angled Triangle ſubtending the Right Angle, is equal to the Figures deſcribed upon the Sides containing the Right Angle, being ſimilar and alike ſituate to the former Figure*; which was to be demonſtrated.

† *Cor.* 20 *of this.*

‡ 24. 5.

PROPOSITION XXXII.

THEOREM.

If two Triangles having two Sides proportional to two Sides, be ſo compounded, or ſet together at one Angle, that their homologous Sides be parallel, then the other Sides of theſe Triangles will be in one ſtrait Line.

LET there be two Triangles ABC, DCE, having two Sides BA, AC, of the one, proportional to to two Sides CD, DE, of the other, *viz.* Let BA be to AC, as CD is to DE; alſo let AB be parallel to DC, and AC to DE. I ſay BC, CE, are both in one ſtrait Line.

For becauſe AB is parallel to DC, and the Right Line AC falls on them, the alternate Angles BAC, ACD, will be * equal to each other; And by the ſame Reaſon, the Angle CDE is equal to the Angle ACD; wherefore the Angle BAC is equal to the Angle CDE. Then becauſe ABC, DCE, are two Triangles, having one Angle A equal to one Angle D, and the Sides about the equal Angles proportional,

* 29. 1.

nal, *viz*. BA to AC, as CD to DE, the Triangle ABC will be * equiangular to the Triangle DCE; wherefore the Angle ABC is equal to the Angle DCE; but the Angle ACD has been proved to be equal to the Angle BAC; therefore the whole Angle ACE is equal to the two Angles ABC, BAC; and if ACB, which is common, be added, then the Angles ACE, ACB, are equal to the Angles BAC, ACB, CBA; but the Angles BAC, ACB, CBA, are equal to two Right Angles. Therefore the Angles ACE, ACB, will alſo be equal to two Right Angles, and ſo at the Point C in the Right Line AC, two Right Lines BC, CE, tending contrary Ways, makes the adjacent Angles ACE, ACB, equal to two Right Angles; therefore BC ſhall be † in the ſame Right Line with CE. Wherefore, *if two Triangles having two Sides proportional to two Sides, be ſo compounded, or ſet together at one Angle, that their homologous Sides be parallel, then the other Sides of theſe Triangles will be in one ſtrait Line*; which was to be demonſtrated.

* 6 of this.

† 14. 1.

PROPOSITION XXXIII.

THEOREM.

In equal Circles the Angles have the ſame Proportion with their Circumferences on which they ſtand, whether the Angles be at the Centers, or at the Circumferences; and ſo likewiſe are the Sectors, as being at the Centers.

LET ABC, DEF, be equal Circles, and let the Angles BGC, EHF, be at their Centers G, H, and the Angles BAC, EDF, at their Circumferences. I ſay, as the Circumference BC is to the Circumference EF, ſo is the Angle BGC to the Angle EHF; and ſo is the Angle BAC to the Angle EDF; and ſo is the Sector BGC to the Sector EHF.

For aſſume any Number of continuous Circumferences CK, KL, each equal to BC; and alſo any Number FM, MN, each equal to EF, and join GK, GL, HM, HN.

Then

Then because the Circumferences BC, CK, KL, are equal to each other, the Angles BGC, CGK,
* 27. 3. KGL, will be * also equal to one another; and so the Circumference BL is the same Multiple of the Circumference BC, as the Angle BGL is of the Angle BGC. For the same Reason, the Circumference NE is the same Multiple of the Circumference EF, as the Angle EHN is of the Angle EHF; but if the Circumference BL be equal to the Circumference EN, then the Angle BGL shall be equal to the Angle EHN; and if the Circumference BL be greater than the Circumference EN, the Angle BGL will be greater than the Angle EHN, and if less, less. Therefore here are four Magnitudes, *viz.* the two Circumferences BC, EF, and the two Angles BGC, EHF; and since there are taken Equimultiples of the Circumference BC, and the Angle BGC; to wit, the Circumference BL, and the Angle BGL; as also Equimultiples of the Circumference EF, and the Angle EHF, *viz.* the Circumference EN, and the Angle EHN. And because it is proved if the Circumference BL exceeds the Circumference EN, the Angle BGL will likewise exceed the Angle EHN; and if equal, equal; if less, less. It shall be as the Cir-
† Def. 5. 5. cumference BC is to the Circumference EF; so is † the Angle BGC to the Angle EHF; but as the An-
‡ 15. 5. gle BGC is to the Angle EHF, so is ‡ the Angle
* 20. 3. BAC to the Angle EDF; for the former are * double to the latter. Therefore as the Circumference BC is to the Circumference EF, so is the Angle BGC to the Angle EHF; and so the Angle BAC to the Angle EDF.

Wherefore in equal Circles, Angles have the same Proportion as the Circumferences they stand on, whether they be at the Centers, or at the Circumferences.

I say, moreover, that as the Circumference BC is to the Circumference EF, so is the Sector GBC to the Sector HFE.

For join BC, CK, and assume the Points X, O, in the Circumferences BC, CK, and join BX, XC, CO, OK.

Then because the two Sides BG, GC, are equal to the two Sides CG, GK, and they contain equal
† 4. 1. Angles, the Base BC shall be † equal to the Base

CK;

CK; as likewise the Triangle GBC to the Triangle GCK. And because the Circumference BC is equal to the Circumference CK, and the Circumference remaining which makes up the whole Circle ABC, is equal to the remaining Circumference which makes up the same Circle, the Angle BXC is equal to the Angle COX; and so the Segment BXC is similar to the Segment COK; and they are upon equal Right Lines BC, CK; but similar Segments of Circles that stand upon equal Right Lines, are * equal to each other: Therefore the Segment BXC is equal to the Segment COK. But the Triangle BGC is also equal to the Triangle CGK; and so the whole Sector BGC will be equal to the whole Sector CGK. By the same Reason, the Sector GKL will be equal to the Sector GBC, or GCK; therefore the three Sectors BGC, CGK, KGL, are equal to one another, so likewise are the Sectors HEF, HFM, HMN. Wherefore the Circumference LB is the same Multiple of the Circumference BC, as the Sector GBL is of the Sector GBC. For the same Reason, the Circumference NE is the same Multiple of the Circumference EF, as the Sector HEN is of the Sector HEF; but if the Circumference BL be equal to the Circumference EN, then the Sector BGL will be equal to the Sector EHN; and if the Circumference BL exceeds the Circumference EN, then the Sector BGL will also exceed the Sector EHN, and if less, less. Therefore since there are four Magnitudes, to wit, the two Circumferences BC, EF, and the two Sectors GBC, EHF; and there are taken of the Circumference BL, and the Sector GBL, Equimultiples of the Circumference BL, and the Sector GBL; as also of the Circumference EN, and the Sector HEN, Equimultiples of the Circumference EF, and the Sector HEF. And because it is proved, that if the Circumference BL exceeds the Circumference EN, the Sector BGL will also exceed the Sector EHN; and if equal, equal, if less, less. Therefore as the Circumference BC is to the Circumference EF, so is the Sector GBC to the Sector HEF; *which was to be demonstrated.*

* 24. 3.

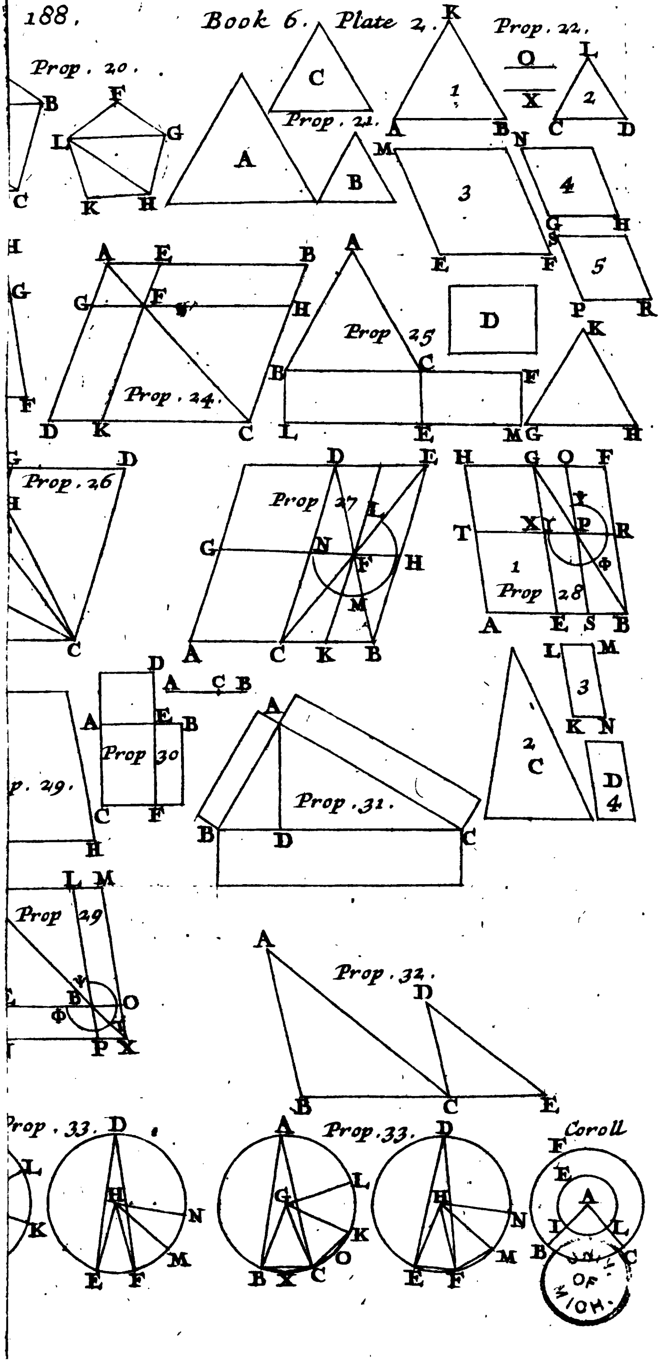
Coroll. 1.

Coroll. 1. An Angle at the Center of a Circle is to four Right Angles, as an Arc on which it ſtands is to the whole Circumference; for as the Angle BAC is to a Right Angle, ſo is the Arc BC to a Quadrant of the Circle: Wherefore if the Conſequents be quadrupled, the Angle BAC ſhall be to four Right Angles, as the Arc BC is to the whole Circumference.

2. The Arc's IL, BC, of unequal Circles, which ſubtend equal Angles, whether at their Centers, or Circumferences, are ſimilar; for IL is to the whole Circumference ILE, as the Angle IAL is to four Right Angles; but as IAL, or BAC, is to four Right Angles, ſo is the Arc BC to the whole Circumference BCF. Therefore as IL is to the whole Circumference ILE, ſo is BC to the whole Circumference BCF; and ſo the Arc's IL, BC, are ſimilar.

3. Two Semi-diameters AB, AC, cut off ſimilar Arc's IL, BC, from concentric Circumferences.

The END *of the* SIXTH BOOK.


Prop. 20.
Prop. 21.
Prop. 22.
Prop. 24.
Prop 25
Prop. 26
Prop 27
Prop 28
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Prop 29
Prop 30
Prop. 31.
Prop. 32.
Prop. 33.
Prop. 33.
Coroll

EUCLID's ELEMENTS.

BOOK XI.

DEFINITIONS.

I. *A Solid is that which has Length, Breadth, and Thickness.*

II. *The Term of a Solid is a Superficies.*

III. *A Right Line is perpendicular to a Plane, when it makes Right Angles with all the Lines that touch it, and are drawn in the said Plane.*

IV. *A Plane is perpendicular to a Plane, when the Right Lines in one Plane, drawn at Right Angles to the common Section of the two Planes, are at Right Angles to the other Plane.*

V. *The Inclination of a Right Line to a Plane, is the acute Angle contained under that Line, and another Right one drawn in the Plane from that End of the inclining Line which is in the Plane, to the Point where a Right Line falls from the other End of the inclining Line perpendicular to the Plane.*

VI. *The Inclination of a Plane to a Plane, is the acute Angle contained under the Right Lines drawn in both the Planes to the same Point of their common Intersection, and making Right Angles with it.*

VII. *Planes are said to be inclined similarly, when the said Angles of Inclination are equal.*

VIII. *Parallel Planes are such, which being produced never meet.*

IX. *Similar solid Figures are such that are contained under equal Numbers of similar Planes.*

X. *Equal and similar solid Figures, are those that are contained under equal Numbers of similar and equal Planes.*

XI. *A solid Angle is the Inclination of more than two Right Lines that touch one another, and are not in the same Superficies: Or, a solid Angle is that which is contained under more than two plane Angles which are not in the same Superficies, but being all at one Point.*

XII. *A Pyramid is a solid Figure comprehended under divers Planes set upon one Plane, and put together at one Point.*

XIII. *A Prism is a solid Figure contained under Planes, whereof the two opposite are equal, similar, and parallel, and the others Parallelograms.*

XIV. *A Sphere is a solid Figure, made when the Diameter of a Semicircle, remaining at rest; the Semicircle is turned about till it returns to the same Place from whence it began to move.*

XV. *The Axis of a Sphere is that fixed Line, about which the Semicircle is turned.*

XVI. *The Center of a Sphere is the same with that of the Semicircle.*

XVII. *The Diameter of a Sphere, is a Right Line drawn thro' the Center, and terminated on either Side by the Superficies of the Sphere.*

XVIII. *A Cone is a Figure described when one of the Sides of a Right-angled Triangle, containing the Right Angle, remaining fixed, the Triangle is turned about till it returns to the Place from whence it first began to move. And if the fixed Right Line be equal to the other that contains the Right Angle, then the Cone is a rectangular Cone; but if it be less, it is an obtused angled Cone; if greater, an acute angled Cone.*

XIX. *The*

XIX. *The Axis of a Cone is that fixed Right Line about which the Triangle is moved.*

XX. *The Base of a Cone is the Circle described by the Right Line mov'd about.*

XXI. *A Cylinder is a Figure described by the Motion of a Right-angled Parallelogram, one of the Sides containing the Right Angle, remaining fixed, while the Parallelogram is turned about to the same Place from whence it begun to be moved.*

XXII. *The Axis of a Cylinder is that fixed Right Line about which the Parallelogram is turned.*

XXIII. *And the Bases of a Cylinder are the Circles that be described by the Motion of the two opposite Sides of the Parallelogram.*

XXIV. *Similar Cones and Cylinders are such, whose Axes and Diameters of their Bases are proportional.*

XXV. *A Cube is a solid Figure contained under six equal Squares.*

XXVI. *A Tetrahedron is a solid Figure contained under four equal equilateral Triangles.*

XXVII. *An Octahedron is a solid Figure contained under eight equal equilateral Triangles.*

XXVIII. *A Dodecahedron is a solid Figure contained under twelve equal equilateral and equiangular Pentagons.*

XXIX. *An Icosahedron is a solid Figure contained under twenty equal equilateral Triangles.*

XXX. *A Parallelepipedon is a Figure contained under six quadrilateral Figures, whereof those which are opposite are parallel.*

PROPOSITION I.

THEOREM.

One Part of a Right Line cannot be in a plane Superficies, and another Part above it.

FOR, if possible, let the Part AB of the Right Line ABC, be in a plane Superficies, and the Part BC above the same.

There will be some Right Line in the aforesaid Plane, which with AB will be but one strait Line. Let this Line be DB.

Then the two given Right Lines ABC, ABD, have one common Segment AB, which is impossible; for one Right Line will not meet another in more Points than one. Wherefore, *one Part of a Right Line cannot be in a plane Superficies, and another Part above it*; which was to be demonstrated.

PROPOSITION II.

THEOREM.

If two Right Lines cut each other, they are both in one Plane; and every Triangle is in one Plane.

LET two Right Lines AB, CD, cut each other in the Point E. I say, they are both in one Plane, and every Triangle is in one Plane.

For take any Points, F and G, in the Right Lines AB, CD, and join CB, FG, and let there be drawn FH, GK. In the first Place, I say, the Triangle EBC is in one Plane.

For if one Part FHC, or GBK, of the Triangle EBC, be in one Plane, and the other Part in another Plane; then one Part of each of the Lines EC, EB, shall be in one Plane, and the other Part in another Plane; which we have proved * to be absurd. Therefore

* 1 of this.

fore the Triangle EBC is in oue Plane, but both the Right Lines EC, EB, are in the same Plane as the Triangle BCE is; and AB, CD, are both in the same Plane as EC, EB are. Wherefore the Right Lines AB, CD, are both in one Plane, and every Triangle is in one Plane; *which was to be demonstrated*.

PROPOSITION III.

THEOREM.

If two Planes cut each other, their common Section will be a Right Line.

LET two Planes AB, CD, cut each other, whose common Section is the Line DB. I say, DB is a Right Line.

For if it be not, draw the Right Line DEB in the Plane AB, from the Point D to the Point B, and the Right Line DFB in the Plane BC.

Then two Right Lines DEB, DFB, have the same Terms, and include a Space, which is * absurd. Therefore DEB, DFB, are not Right Lines. In the same Manner we demonstrate, that no other Line drawn from the Point D to the Point B, is a Right Line, besides DB, the common Section of the Planes AB, BC. *If*, therefore, *two Planes cut each other, their common Section will be a Right Line*; which was to be demonstrated.

* *Axiom*.

PROPOSITION IV.

THEOREM.

If to two Right Lines, cutting one another, a third stands at Right Angles in the common Section, it shall be also at Right Angles to the Plane drawn thro' the said Lines.

LET the Right Line EF stand at Right Angles to the two Right Lines AB, CD, in the common Section E. I say, EF is also at Right Angles to the Plane drawn thro' AB, CD.

For take the Right Lines EA, EB, CE, DE, equal, and thro' E any how draw the Right Line GEH, and join AD, CB; and from the Point F let there be drawn FA, FG, FD, FC, FH, FB: Then because two Right Lines AE, ED, are equal
* 15. 1. to two Right Lines CE, EB, and they contain * the
† 4. 1. equal Angles AED, CEB; the Base AD shall be † equal to the Base CB, and the Triangle AED equal to the Triangle CEB; and so likewise is the Angle DAE equal to the Angle EBC; but the Angle AEG is * equal to the Angle BEH; therefore AGE, BEH, are two Triangles, having two Angles of the one equal to two Angles of the other, each to each, and one Side AE equal to one Side EB, *viz.* those that are at the equal Angles; and so the other Sides
‡ 26. 1. of the one, will be ‡ equal to the other Sides of the other. Therefore GH is equal to EH, and AG to BH; and since AE is equal to EB, and FE is common and at Right Angles, the Base AF shall be † equal to the Base FB: For the same Reason likewise, shall CF be equal to FD. Again, because AD is equal to CB, and AF to FB, the two Sides FA, AD, will be equal to the two Sides FB, BC, each to each; but the Base DF has been proved equal to
‖ 8. 1. the Base FC: Therefore the Angle FAD is ‖ equal to the Angle FBC: Moreover, AG has been proved equal to BH; but FB also is equal to AF. Therefore the two Sides FA, AG, are equal to the two Sides FB, BH; and the Angle FAG is equal to the Angle FBH, as has been demonstrated; wherefore the Base GF is † equal to the Base FH. Again, because GE has been proved equal to EH, and EF is common, the two Sides GE, EF, are equal to the two Sides HE, EF; but the Base HF is equal to the Base FG; therefore the Angle GEF is ‖ equal to the Angle HEF, and so both the Angles GEF, HEF, are Right Angles: Therefore FE makes Right Angles with GH, which is any how drawn thro' E. After the same Manner we demonstrate that FE is at Right Angles to all Right Lines that are drawn in
* Def. 3 of this. the Plane to it; but a Right Line is * at Right Angles to a Plane, when it is at Right Angles to all Right Lines drawn to it in the Plane. Therefore FE is at Right Angles to a Plane drawn thro' the Right Lines

AB,

AB, CD. Wherefore, *if to two Right Lines cutting one another, a third stands at Right Angles in the common Section, it shall be also at Right Angles to the Plane drawn thro' the said Lines*; which was to be demonstrated.

PROPOSITION V.

THEOREM.

If to three Right Lines, touching one another, a third stands at Right Angles in their common Section, those three Right Lines shall be in one and the same Plane.

LET the Right Line AB stand at Right Angles in the Point of Contact B, to the three Right Lines BC, BD, BE. I say BC, BD, BE, are in one and the same Plane.

For if they are not, let BD, BE, be in one Plane, and BC above it; and let the Plane passing thro' AB, BC, be produced, and it will * make the common Section, with the other Plane, a strait Line, which let be BF. Then three Right Lines AB, BC, BF, are in one Plane drawn thro' AB, BC; and since AB stands at Right Angles to BD and BE, it shall be † at Right Angles to a Plane drawn thro' BE, DB; and so AB shall make ‡ Right Angles with all Right Lines touching it that are in the same Plane; but BF being in the said Plane, touches it. Wherefore the Angle ABF is a Right Angle, but the Angle ABC (by the *Hyp.*) is also a Right Angle. Therefore the Angle ABF is equal to the Angle ABC, and they are both in the same Plane, which cannot be; and so the Right Line BC is not above the Plane passing thro' BE and BD. Wherefore the three Lines BC, BD, BE, are in one and the same Plane. Therefore, *if to three Right Lines, touching one another, a third stands at Right Angles in their common Section, those three Right Lines shall be in one and the same Plane*; which was to be demonstrated.

* 3 *of this.*

† 4 *of this.*

‡ *Def.* 3.

PROPOSITION VI.

THEOREM.

If two Right Lines be perpendicular to one and the same Plane, those Right Lines are parallel to one another.

LET two Right Lines AB, CD, be perpendicular to one and the same Plane. I say, AB is parallel to CD.

For let them meet the Plane in the Points B, D, and join the Right Line BD, to which let DE be drawn in the same Plane at Right Angles; make DE equal to AB, and join BE, AE, AD.

Then because AB is at Right Angles to the aforesaid Plane, it shall be * at Right Angles to all Right Lines, touching it, drawn in the Plane; but AB touches BD, BE, which are in the said Plane. Therefore each of the Angles ABD, ABE, is a Right Angle. So for the same Reason likewise, is each of the Angles CDB, CDE, a Right Angle. Then because AB is equal to DE, and BD is common, the two Sides AB, BD, shall be equal to the two Sides ED, DB; but they contain Right Angles. Therefore the Base AD is † equal to the Base BE. Again, because AB is equal to DE, and AD to BE, the two Sides AB, BE, are equal to the two Sides ED, DA; but AE, their Base, is common. Wherefore the Angle ABE is ‡ equal to the Angle EDA; but ABE is a Right Angle. Therefore EDA is also a Right Angle; and so ED is perpendicular to DA; but it is also perpendicular to BD and DC. Therefore ED is at Right Angles in the Point of Contact to three Right Lines BD, DA, DC. Wherefore these three last Right Lines are * in one Plane: But BD, DA, are in the same Plane as AB is; for every Triangle is † in the same Plane. Therefore it is necessary that AB, BD, DC, be in one Plane; but both the Angles ABD, BDC, are Right Angles Wherefore AB is ‡ parallel to CD. Therefore, *if two Right Lines be perpendicular to one and the same Plane, those Right Lines are parallel to one another*; which was to be demonstrated.

* Def. 3 of this.

† 4. 1.

‡ 8. 1.

* 5 of this.

† 2 of this.

‡ 28. 1.

PRO-

PROPOSITION VII.

THEOREM.

If there be two Parallel Lines, and any Points be taken in both of them, the Right Lines joining those Points shall be in the same Plane as the Parallels are.

LET AB, CD, be two parallel Right Lines, in which are taken any Points E, F. I say, a Right Line joining the Points E, F, are in the same Plane as the Parallels are.

For if it be not, let it be elevated above the same, if possible, as EGF; thro' which let some Plane be drawn; whose Section, with the Plane in which the Parallels are, let * be the Right Line EF, then the two Right Lines EGF, EF, will include a Space, which is † absurd. Therefore a Right Line, drawn from the Point E to the Point F, is not elevated above the Plane, and consequently it must be in that passing thro' the Parallels AB, CD. Wherefore, *if there be two parallel Lines, and any Points be taken in both of them, the Right Line joining these Points shall be in the same Plane as the Parallels are*; which was to be demonstrated.

* 3 *of this.*

† *Axiom* 10. 1.

PROPOSITION VIII.

THEOREM.

If there be two parallel Right Lines, one of which is perpendicular to some Plane, then shall the other be perpendicular to the same Plane.

LET AB, CD be two parallel Right Lines, one of which, as AB is perpendicular to some Plane. I say, the other CD is also perpendicular to the same Plane.

See the Fig. of Prop. VI.

For let AB, CD, meet the Plane in the Points B, D, and let BD be joined; then AB, CD, BD, are * in one Plane. Let DE be drawn in the Plane at Right Angles to BD, and make DE equal to AB, and join BE, AE, AD. Then since AB is perpendicular

* 7 *of this.*

* *Def.* 3 dicular to the Plane, it will * be perpendicular to all Right Lines, touching it drawn in the same Plane; therefore each of the Angles ABD, ABE, is a Right Angle. And since the Right Line BD falls on the Right Lines AB, CD, the Angles ABD, CDB,
† 29. 1. shall be † equal to two Right Angles. Therefore the Angle CDB is also a Right Angle, and so CD is perpendicular to DB: And since AB is equal to DE, and BD is common, the two Sides AB, BD, are equal to the two Sides ED, DB. But the Angle ABD is equal to the Angle EDB; for each of them
‡ 4. 1. is a Right Angle. Therefore the Base AD is ‡ equal to the Base BE. Again, since AB is equal to DE, and BE to AD, the two Sides AB, BE, shall be equal to the two Sides ED, DA, each to each; but the Base AE is common. Wherefore the Angle ABE
* 8. 1. is * equal to the Angle EDA; but the Angle ABE is a Right Angle. Therefore EDA is also a Right Angle, and ED is pergendicular to DA; but it is likewise perpendicular to DB: Therefore ED shall also
† 4 *of this.* be † perpendicular to the Plane passing thro' BD, DA,
‡ *Def.* 3. and likewise shall be ‡ at Right Angles to all Right Lines, drawn in the said Plane that touch it. But DC is in the Plane passing thro' BD, DA, because
* 2 *of this.* AB, BD, are * in that Plane; and DC is † in the
† 7 *of this.* same Plane that AB and BD are in. Wherefore ED is at Right Angles to DC, and so CD is at Right Angles to DE, as also to DB. Therefore CD stands at Right Angles in the common Section D, to two Right Lines DE, DB, mutually cutting one another; and accordingly is at Right Angles to the Plane passing thro' DE, DB; *which was to be demonstrated.*

PRO-

PROPOSITION IX.

THEOREM.

Right Lines that are parallel to the same Right Line, not being in the same Plane with it, are also parallel to each other.

LET both the Right Lines AB, CD, be parallel to the Right Line EF, not being in the same Plane with it. I say, AB is parallel to CD.

For assume any Point G in EF, from which Point G, let GH be drawn at Right Angles to EF, in the Plane passing thro' EF, AB: Also let GK be drawn at Right Angles to EF in the Plane passing thro' EF, CD: Then because EF is perpendicular to GH, and GK, the Line EF shall also be * at Right Angles to a Plane passing thro' GH, GK; but EF is parallel to AB. Therefore AB is † also at Right to the Plane passing thro' HGK. For the same Reason, CD is also at Right Angles to the Plane passing thro' HGK; and therefore AB and CD, will be both at Right Angles to the Plane passing thro' HGK. But if two Right Lines be at Right Angles to the same Plane, they shall be ‡ parallel to each other. Therefore AB is parallel to CD; *which was to be demonstrated.*

* *4 of this.* † *8 of this.* ‡ *6 of this.*

PROPOSITION X.

THEOREM

If two Right Lines, touching one another, be parallel to two other Right Lines, touching one another, but not in the same Plane, these Right Lines contain equal Angles.

LET two Right Lines AB, BC, touching one another, be parallel to two Right Lines DE, EF, touching one another, but not in the same Plane. I say, the Angle ABC is equal to the Angle DEF.

For take BA, BC, ED, EF, equal to one another, and join AD, CF, BE, AC, DF: Then because BA is equal and parallel to ED, the Right Line

* 31. 1. AD shall also be * equal and parallel to BE. For the same Reason, CF will be equal and parallel to BE; therefore AD, CF, are both equal and parallel to BE. But Right Lines that are parallel to the same Right Line, not being in the same Plane with it, will
† 9 of this. be † parallel to each other. Therefore AD is parallel and equal to CF, but AC, DF, joins them; where-
‡ 33. 1. fore AC is ‡ equal and parallel to DF. And because two Right Lines AB, BC, are equal to two Right Lines DE, EF, and the Base AC equal to the Base
* 8. 1. EF, the Angle ABC will be * equal to the Angle DEF. Therefore, *if two Right Lines, touching one another, be parallel to two other Right Lines, touching one another, but not in the same Plane, those Right Lines contain equal Angles*; which was to be demonstrated.

PROPOSITION XI.

PROBLEM.

From Point given above a Plane, to draw a Right Line perpendicular to that Plane.

LET A be a Point given above the given Plane BH. It is requir'd to draw a Right Line from the Point A, perpendicular to the Plane BH.

Let a Right Line BC be any how drawn in the
* 12. 1. Plane BH, and let AD be drawn * from the Point A perpendicular to BC; then if AD be perpendicular to the Plane BH, the Thing required is already done. But if not, let DE be drawn in the Plane from the Point D at Right Angles to BC; and let AF be drawn * from the Point A perpendicular to DE. Lastly, thro' F draw GH parallel to BC.

Then because BC is perpendicular to both DA and
† 14 of this. DE, BC will also be † perpendicular to a Plane passing thro' ED, DA. But GH is parallel to BG. And if there are two Right Lines parallel, one of which is a Right Angles to some Plane, then shall
‡ 8 of this. the other be ‡ at Right Angles to the same Plane. Wherefore GH is at Right Angles to the Plane passing
* Def. 3. thro' ED, DA, and so is * perpendicular to all the Right Lines in the same Plane that touch it. But AF, which

which is in the Plane passing thro' ED and DA, doth touch it. Therefore GH is perpendicular to AF, and so AF is perpendicular to GH; but AF likewise is perpendicular to DE; therefore AF is perpendicular to both HG, DE. But if a Right Line stands at Right Angles to two Right Lines, in their common Section, that Line will be † at Right Angles to the Plane passing thro' these Lines. Therefore AF is perpendicular to the Plane drawn thro' ED, GH; that is, to the given Plane BH. Therefore AF is drawn from the given Point A, above the given Plane BH, perpendicular to the said Plane; *which was to be done.* † 4 *of this.*

PROPOSITION XII.

PROBLEM.

To erect a Right Line perpendicular to a given Plane, from a Point given therein.

LET A be a given Point in a given Plane MN. It is requir'd to draw a Right Line from the Point A, at Right Angles, to the Plane MN.

Let some Point B be supposed above the given Plane, from which let BC be drawn * perpendicular to the said Plane; and let AD be drawn † from A parallel to BC. * 11 *of this.* † 31. 1.

Then because AD, CB, are two parallel Right Lines, one of which, *viz.* BC, is perpendicular to the Plane MN; the other AD shall be ‡ also perpendicular to the same Plane. Therefore, *a Right Line is erected perpendicular to a given Plane, from a Point given therein*; which was to be done. ‡ 8 *of this.*

PRO-

PROPOSITION XIII.

THEOREM.

Two Right Lines cannot be erected at Right Angles, to a given Plane from a Point therein given.

FOR if it is possible, let two Right Lines AB, AC, can be erected perpendicular to a given Plane on the same Side, at a given Point A, in a given Plane.

Then let a Plane be drawn thro' BA, AC, cutting the given Plane thro' A in the Right Line * DAE; therefore the Right Lines AB, AC, DAE, are in one Plane. And because CA is perpendicular to the given Plane, it shall also be † perpendicular to all Right Lines drawn in that Plane, and touching it; but DAE being in the given Plane touches it. Therefore the Angle CAE is a Right Angle. For the same Reason, BAE is also a Right Angle; wherefore the Angle CAE is equal to BAE, and they are both in one Plain, which is absurd. Therefore, *two Right Lines cannot be erected at Right Angles, to a given Plane, from a Poiut therein given*; which was to be demonstrated.

* 3 of this.

† Def. 3.

PROPOSITION XIV.

THEOREM.

Those Planes, to which the same Right Line is perpendicular, are parallel to each other.

LET the Right Line AB be perpendicular to each of the Planes CD, EF. I say, these Planes are parallel.

For if they be not, let them be produced till they meet each other, and let the Right Line GH be the common Section, in which take any Point K, and join AK, BK. Then because AB is perpendicular to the Plane EF, it shall also be perpendicular to the Right Line BK, being in the Plane EF produced. Wherefore the Angle ABK is a Right Angle. And for the same Reason, BAK is also a Right Angle.

And

And so the two Angles ABK, BAK, of the Triangle ABK, are equal to two Right Angles, which is * impossible. Therefore the Planes CD, EF, being produced, will not meet each other, and so are necessarily parallel. Therefore, *those Planes, to which the same Right Line is perpendicular, are parallel to each other*; which was to be demonstrated. * 17. 1.

PROPOSITION XV.

THEOREM.

If two Right Lines, touching one another, be parallel to two Right Lines, touching one another, and not being in the same Plane with them, the Planes drawn thro' those Right Lines are parallel to each other.

LET two Right Lines AB, BC, touching one another, be parallel to two Right Lines DE, EF, touching one another, but not in the same Plane with them. I say, the Planes passing thro' AB, BC, and DE, EF, being produced, will not meet each other.

For let BG be drawn from the Point B, perpendicular to the Plane passing thro' DE, EF, meeting the same in the Point G; and thro' G let GH be drawn parallel to ED, and GK parallel to EF; then because BG is perpendicular to the Plane passing thro' DE, EF; it shall also make * Right Angles with all Right Lines that touch it, and are in the same Plane; but GH and GK, which are both in the same Plane, touch it. Therefore each of the Angles BGH, BGK, is a Right Angle. And since BA is parallel to GH, the Angles GBA, GBH, are † equal to the Right Angles: But BGH is a Right Angle; wherefore GBA shall also be a Right Angle, and so BG is perpendicular to BA. For the same Reason, GB is also perpendicular to BC. Therefore since a Right Line BG, stands at Right Angles to two Right Lines BA, BC, mutually cutting each other; BG shall also be ‡ at Right Angles to the Plane drawn thro' BA, BC. But it is perpendicular to the Plane drawn thro' DE, EF; therefore BG is perpendicular to both the Planes drawn thro' AB, BC, and DE, EF. But those Planes to which the same Right Line

* Def. 3.
† 29. 1.
‡ 4 of this.

Line

* 14 of this. Line is perpendicular, are * parallel. Therefore the Plane drawn thro' AB, BC, is parallel to the Plane drawn thro' DE, EF. Wherefore, *if two Right Lines, touching one another, be parallel to two Right Lines, touching one another, and not being in the same Plane with them, the Planes drawn thro' these Right Lines are parallel to each other.*

PROPOSITION XVI.

THEOREM.

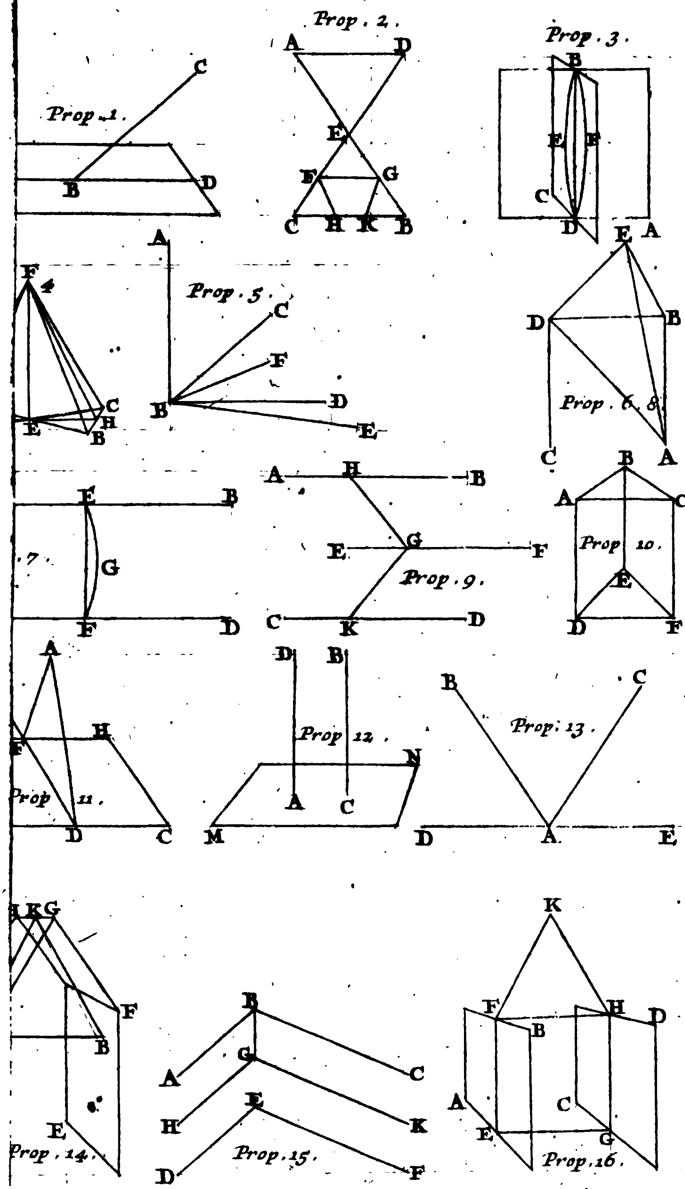
If two parallel Planes are cut by any other Plane, their common Sections will be parallel.

LET two parallel Planes, AB, CD, be cut by any Plane EFHG, and let their common Sections be EF, GH. I say EF is parallel to GH.

For if it is not parallel, EF, GH, being produc'd, will meet each other either on the Side FH, or EG. First let them be produced on the Side FH, and meet in K; then because EFK is in the Plane AB, all Points taken in EFK will be in the same Plane. But K is one of the Points that is in EFK. Therefore K is in the Plane AB. For the same Reason K is also in the Plane CD. Wherefore the Planes AB, CD, will meet each other. But they do not meet, since they are suppos'd parallel. Therefore the Right Lines EF, GH, will not meet on the Side FH. After the same Manner it is prov'd, that they will not meet, if produc'd, on the Side EG. But Right Lines, that will neither Way meet each other, are parallel; therefore EF is parallel to GH. *If*, therefore, *two parallel Planes are cut by any other Plane, their common Sections will be parallel*; which was to be demonstrated.

PRO-

Book. 11. Plate. 1.



PROPOSITION XVII.

THEOREM.

If two Right Lines are cut by parallel Planes, they shall be cut in the same Proportion.

LET two Right Lines AB, CD, be cut by Parallel Planes GH, KL, MN, in the Points A, E, B, C, F, D. I say, as the Right Line AE is to the Right Line EB, so is CF to FD.

For let AC, BD, AD, be join'd: Let AD meet the Plane KL in the Point X, and join EX, XF. Then because two parallel Planes KL, MN, are cut by the Plane EBDX, their common Sections EX, BD, are * parallel. For the same Reason, because * 16 of this. two parallel Planes GH, KL, are cut by the Plane AXFC, their common Sections AC, FX, are parallel; and because EX is drawn parallel to the Side BD of the Triangle ABD, it shall be as AE is to EB, so is † AX to XD. Again, because XF is † 2. 6. drawn parallel to the Side AC of the Triangle ADC, it shall be † as AX is to XD, so is CF to FD. But it has been prov'd, as AX is to XD, so is AE to EB. Therefore, as AE is to EB, so is ‡ CF to FD. ‡ 11. 5. Wherefore, *if two Right Lines are cut by parallel Planes, they shall be cut in the same Proportion*; which was to be demonstrated.

PROPOSITION XVIII.

THEOREM.

If a Right Line be perpendicular to some Plane, then all Planes passing thro' that Line will be perpendicular to the same Plane.

LET the Right Line AB be perpendicular to the Plane CL. I say, all Planes that pass thro' AB, are likewise perpendicular to the Plane CL.

For let a Plane DE pass thro' the Right Line AB, whose common Section, with the Plane CL, is the Right Line CE; and take some Point F in CE; from which

which let FG be drawn in the Place DE, perpendicular to the Right Line CE. Then because AB is perpendicular to the Plane CL, it shall also be * perpendicular to all the Right Lines which touch it, and are in the same Plane. Wherefore it is perpendicular to CE; and consequently the Angle ABF is a Right Angle; but GFB is likewise a Right Angle. Therefore AB is parallel to FG. But AB is at Right Angles to the Plane CL. Therefore FG will be † at Right Angles to that same Plane. But one Plane is perpendicular to another, when the Right Lines, drawn in one of the Planes perpendicular to the common Section of the Planes, are ‡ perpendicular to the other Plane. But FG is drawn in one Plane DE, perpendicular to the common Section CE of the Planes. And it has been prov'd to be perpendicular to the Plane CL. Therefore the Plane DE is at Right Angles to the Plane CL. After the same Manner it is demonstrated, that all Planes, passing thro' the Right Line AB, are perpendicular to the Plane CL. Therefore, *if a Right Line be perpendicular to some Plane, then all Planes passing thro' that Line will be perpendicular to the same Plane*; which was to be demonstrated.

* Def. 3.

† 8 of this.

‡ Def. 4. of this.

PROPOSITION. XIX.

THEOREM.

If two Planes, cutting each other, be perpendicular to some Plane, then their common Section will be perpendicular to that same Plane.

LET two Planes AB, BC, cutting each other, be perpendicular to some third Plane, and let their common Section be BD. I say, BD is perpendicular to the said third Plane, which let be ADC.

For, if possible, let BD not be perpendicular to the third Plane; and from the Point D, let DE be drawn in the Plane AB, perpendicular to AD; and let DF be drawn in the Plane BC, perpendicular to CD; then because the Plane AB is perpendicular to the third Plane, and DE is drawn in the Plane AB, perpendicular to their common Section AD, DE shall

ſhall be * perpendicular to the third Plane. In like manner we prove, that DF alſo is perpendicular to the ſaid Plane. Wherefore two Right Lines ſtand at Right Angles, to this third Plane, on the ſame Side at the ſame Point D; which is † abſurd. Therefore to this third Plane cannot be erected any Right Lines perpendicular at D, and on the ſame Side, except BD, the common Section of the Planes AB, BC. Wherefore DB is perpendicular to the third Plane. *If*, therefore, *two Planes, cutting each other, be perpendicular to ſome Plane, then their common Section will be perpendicular to that ſame Plane*; which was to be demonſtrated. * *Def.* 4. † 13 *of this.*

PROPOSITION XX.

THEOREM.

If a ſolid Angle be contained under three plain Angles, any two of them, howſoever taken, are greater than the third.

LET the ſolid Angle A be contained under three plain Angles BAC, CAD, DAB. I ſay any two of the Angles BAC, CAD, DAB, are greater than the third, howſoever taken.

For if the Angles BAC, CAD, DAB, be equal, it is evident that any two, howſoever taken, are greater than the third. But if not, let BAC be the greater; and make* the Angle BAE, at the Point A, with the Right Line AB, in a Plane paſſing thro' BA, AC, equal to the Angle DAB, make AE equal to AD; thro' E draw BEC, cutting the Right Lines AB, AC, in the Points B, C, and join DB, DC. Then becauſe DA is equal to AE, and AB is common, the two Sides DA, AB, are equal to the two Sides AE, AB; but the Angle DAB is equal to the Angle BAE. Therefore the Baſe DB is † equal to the Baſe BE. And ſince the two Sides DB, DC, are greater than BC, and DB has been prov'd equal to BE, the remaining Side DC ſhall be greater than the remaining Side EC; and ſince DA is equal to AE, and AC is common, and the Baſe DC greater than the Baſe EC, the Angle DAC ſhall be ‡ greater than * 23. 1. † 4. 1. ‡ 25. 1.

than the Angle EAC. But from Construction, the Angle DAB is equal to the Angle BAE. Wherefore the Angles DAB, DAC, are greater than the Angle BAC. After this Manner we demonstrate, if any two other Angles be taken, that they are greater than the third. Therefore, *if a solid Angle be contain'd under three plain Angles, any two of them, howsoever taken, are greater than the third*; which was to be demonstrated.

PROPOSITION XXI.

THEOREM.

Every solid Angle is contain'd under plane Angles together less than four Right ones.

LET A be a solid Angle, contain'd under plane Angles BAC, CAD, DAB. I say the Angles BAC, CAD, DAB, are less than four Right Angles.

For take any Points B, C, D, in each of the Lines AB, AC, AD, and join BC, CD, DB. Then because the solid Angle at B is contain'd under three plane Angles CBA, ABD, CBD, any two of these are * greater than the third. Therefore the Angles CBA, ABD, are greater than the Angle CBD. For the same Reason, the Angles BCA, ACD, are greater than the Angle BCD; and the Angles CDA, ADB, greater than the Angle CDB. Wherefore the six Angles CBA, ABD, BCA, ACD, ADC, ADB, are greater than the three Angles CBD, BCD, CDB. But the three Angles CBD, BCD, CDB, are † equal to two Right Angles. Wherefore the six Angles CBA, ABD, BCA, ACD, ADC, ADB, are greater than two Right Angles. And since the three Angles of each of the Triangles ABC, ACD, ADB, are equal to two Right Angles. The nine Angles of those Triangles CBA, BCA, BAC, ACD, CAD, ADC, ADB, ABD, DAB, are equal to six Right Angles. Six of which Angles CBA, BCA, ACD, ADC, ADB, ABD, are greater than two Right Angles. Therefore the three other Angles BAC, CAD, DAB, which contain the solid Angle,

* 20 of this.

† 32. 1.

gle, will be less than four Right Angles: Wherefore *every solid Angle is contain'd under Angles together, less than four plane Right ones*; which was to be demonstrated.

PROPOSITION XXII.

THEOREM.

If there be three plane Angles, whereof two, any how taken, are greater than the third, and the Right Lines that contain them be equal; then it is possible to make a Triangle of the Right Lines joining the equal Right Lines, which form the Angles.

LET ABC, DEF, GHK, be given plane Angles, any two whereof are greater than the third; and let the equal Right Lines AB, BC, DE, EF, GH, HK, contain them; and let AC, DF, GK, be join'd. I say, it is possible to make a Triangle of AC, DF, GK, that is, any two of them, howsoever taken, are greater than the third.

For if the Angles at B, E, H, are equal, then AC, DF, GK, will be * equal, and any two of them greater than the third; but if not, let the Angles at B, E, H, be unequal, and let the Angle B be greater than either of the others at E or H. Then the Right Line AC will be † greater than either DF or GK; and it is manifest, that AC, together with either DF, or GK, is greater than the other. I say likewise, that DF, GK together, are greater than AG. For make ‡ at the Point B, with the Right Line AB, the Angle ABL, equal to the Angle GHK; and make BL equal to either AB, BC, DE, EF, GH, HK, and join AL, CL. Then, because the two Sides AB, BL, are equal to the two Sides GH, HK, each to each, and they contain equal Angles, the Base AL shall be equal to the Base GK. And since the Angles E and H are greater than the Angle ABC, whereof the Angle GHK is equal to the Angle ABL, the other Angle at E shall be greater than the Angle LBC. And since the two Sides LB, BC, are equal to the two Sides DE, EF, each to each, and the Angle DEF is greater than the Angle LBC, the

* 4. 1.

† 24. 1.

‡ 23. 1.

* 24. 1. Base DF shall be * greater than the Base LC. But GK has been prov'd equal to AL. Therefore DF, GK, are greater than AL, LC; but AL, LC, are greater than AC. Wherefore DF, GK, shall be much greater than AC. Therefore any two of the Right Lines AC, DF, GK, howsoever taken, are greater than the other: And so a Triangle may be made of AC, DF, GK; *which was to be demonstrated.*

PROPOSITION XXIII.

PROBLEM.

To make a solid Angle of three plane Angles, whereof any two, howsoever taken, are greater than the third; but these three Angles must be less than four Right Angles.

LET ABC, DEF, GHK, be three plane Angles given, whereof any two, howsoever taken, are greater than the other, and let the said three Angles be less than four Right Angles. It is requir'd to make a solid Angle of three plane Angles equal to ABC, DEF, GHK.

Let the Right Lines AB, BC, DE, EF, GH, HK, be cut off equal, and join AC, DF, GK; then it is
* 22 of this. possible to make * a Triangle of three Right Lines
† 22. 1. equal to AC, DF, GK: And so let † the Triangle LMN be made, so that AC be equal to LM, and DF to MN, and GK to LN; and let the Circle
‡ 5. 4. LMN be describ'd ‡ about the Triangle, whose Center let be X, which will be either within the Triangle LMN, or on one Side thereof, or without the same.

First let it be within, and join LX, MX, NX. I say AB is greater than LX. For if this be not so, AB shall be either equal to LX, or less. First let it be equal; then because AB is equal to LX, and also to BC, LX shall be equal to BC; but LX is equal to XM. Therefore the two Sides AB, BC, are equal to the two Sides LE, XM, each to each; but the Base AC is put equal to the Base LM. Where-
* 8. 1. fore the Angle ABC shall be * equal to the Angle LXM

LXM. For the ſame Reaſon, the Angle DEF is equal to the Angle MXN, and the Angle GHK to the Angle NXL. Therefore the three Angles ABC, DEF, GHK, are equal to the three Angles LXM, MXN, NXL. But the three Angles LXM, MXN, NXL, are * equal to four Right Angles: And ſo the three Angles ABC, DEF, GHK, ſhall alſo be equal to four Right Angles; but they are put leſs than four Right Angles, which is abſurd. Therefore AB is not equal to LX. I ſay alſo it is neither leſs than LX; for if this be poſſible, make XO equal to AB, and XP to BC, and join OP. Then becauſe AB is equal to BC, XO ſhall be equal to XP; and the remaining Part OL equal to the remaining Part PM: And ſo LM is † parallel to OP, and the Triangle LMX is equiangular to the Triangle OPX. Wherefore XL is ‡ to LM, as XO is to OP; and (by Alternation) as XL is to XO, ſo is LM to OP. But LX is greater than XO. Therefore LM ſhall alſo be greater than OP But LM is put equal to AC. Wherefore AC ſhall be greater than OP. And ſo becauſe the two Right Lines AB, BC, are equal to the two Right Lines OX, XP, and the Baſe AC greater than the Baſe OP; the Angle ABC will be * greater than the Angle OXP. In like manner, we demonſtrate that the Angle DEF is greater than the Angle MXN, and the Angle GHK, than the Angle NXL. Therefore the three Angles ABC, DEF, GHK, are greater than the three Angles LXM, MXN, NXL. But the Angles ABC, DEF, GHK, are put leſs than four Right Angles. Therefore the Angles LXM, MXN, NXL, ſhall be leſs by much than four Right Angles, and alſo equal † to four Right Angles; which is abſurd. Wherefore AB is not leſs than LX. It has alſo been prov'd not to be equal to it. Therefore it muſt neceſſarily be greater. On the Point X raiſe ‡ XR, perpendicular to the Plane of the Circle LMN; whoſe Length let be ſuch, that the Square thereof be equal to the Exceſs, by which the Square of AB exceeds the Square of LX; and let RL, RM, RN, be join'd. Becauſe RX is perpendicular to the Plane of the Circle LMN, it ſhall alſo be * perpendicular to LX, MX, NX. And becauſe LX is equal to

* Cor. 15. 1.
† 2. 6.
‡ 4. 6.
* 25. 1.
† Cor. 15. 1.
‡ 12 of this
* Def. 3.

XM, and XR is common, and at Right Angles to
* 4. 1. them, the Base LR shall be * equal to the Base RM. For the same Reason, RN is equal to RL, or RM. Therefore three Right Lines, RL, RM, RN, are equal to each other. And because the Square of XR is equal to the Excess, by which the Square of AB exceeds the Square of LX; the Square of AB will be equal to the Squares of LX, XR together. But the
† 47. 1. Square of RL is † equal to the Squares of LX, XR: For LXR is a Right Angle. Therefore the Square of AB will be equal to the Square of RL; and so AB is equal to RL. But BC, DE, EF, GH, HK, are every of them equal to AB; and RN, or RM, equal to RL. Wherefore AB, BC, DE, EF, GH, HK, are each equal to RL, RM, or RN: And since the two Sides RL, RM, are equal to the two Sides AB, BC, and the Base LM is put equal to the
‡ 8. 1. Base AC, the Angle LRM shall be ‡ equal to the Angle ABC. For the same Reason the Angle MRN is equal to the Angle DEF, and the Angle LRN equal to the Angle GHK. Therefore a solid Angle is made at R of three plane Angles LRM, MRN, LRN, equal to three plane Angles given, ABC, DEF, GHK.

Now let the Center of the Circle X be in one Side of the Triangle, *viz.* in the Side MN, and join XL. I say again, that AB is greater than LX. For if it be not so, AB will be either equal, or less than LX. First let it be equal, then the two Sides AB, BC, are equal to the two Sides MX, LX, that is, they are equal to MN; but MN is put equal to DF.
* 20. 1. Therefore DE, EF, are equal to DF, which is * impossible. Therefore AB is not equal to LX. In like manner, we prove that it is neither lesser; for the Absurdity will much more evidently follow. Therefore AB is greater than LX. And if in like manner, as before, the Square of RX be made equal to the Excess, by which the Square of AB exceeds the Square of LX, and RX be raised at Right Angles to the Plane of the Circle, the Problem will be done.

Lastly, let the Center X of the Circle be without the Triangle LMN, and join LX, MX, NX. I say AB is greater than LX. For if it be not, it

must

must either be equal or less. First, let it be equal; then the two Sides AB, BC, are equal to the two Sides MX, XL, each to each; and the Base AC is equal to the Base ML; therefore the Angle ABC is equal to the Angle MXL. For the same Reason, the Angle GHK is equal to the Angle LXN; and so the whole Angle MXN, is equal to the two Angles ABC, GHK; but the Angles ABC, GHK, are greater than the Angle DEF. Therefore the Angle MXN is greater than DEF; but because the two Sides DE, EF, are equal to the two Sides MX, XN, and the Base DF is equal to the Base MN, the Angle MXN shall be equal to the Angle DEF; but it has been proved greater, which is absurd. Therefore AB is not equal to LX. Moreover we will prove that it is not less; wherefore it shall be necessarily greater. And if, again, XR be raised at Right Angles to the Plane of the Circle, and made equal to the Side of that Square, by which the Square of AB exceeds the Square of LX, the Problem will be determined. Now, I say, AB is not less than LX; for if it is possible that it can be less, make XO equal to AB, and XP equal to BC, and join OP. Then, because AB is equal to BC, XO shall be equal to XP, and the remaining Part OL equal to the remaining Part PM; therefore LM is * parallel to PO, and the Triangle LMX equiangular to the Triangle PXO. Wherefore as † XL is to LM, so is XO to OP: And (by Alternation) as LX is to XO, so is LM to OP; but LX is greater than XO; therefore LM is greater than OP; but LM is equal to AC; wherefore AC shall be greater than OP. And so because the two Sides AB, BC, are equal to the two Sides OX, XP, each to each; and the Base AC is greater than the Base OP; the Angle ABC shall be ‡ greater than the Angle OXP. So likewise if XR be taken equal to XO or XP, and OR be joined, we prove that the Angle GHK is greater than the Angle OXR. At the Point X, with the Right Line LX, make the Angle LXS equal to the Angle ABC, and the Angle LXT equal to the Angle GHK, and XS, XT, each equal to XO, and join OS, OT, ST. Then because the two Sides AB, BC, are equal to the two Sides OX, XS, and

* 2. 6.

† 4. 6.

‡ 25. 1.

the Angle ABC is equal to the Angle OXS, the Base AC; that is, LM shall be equal to the Base OS. For the same Reason, LN is also equal to OT. And since the two Sides ML, LN, are equal to the two Sides OS, OT, and the Angle MLN greater than the Angle SOT; the Base MN shall be greater than the Base ST; but MN is equal to DF; therefore DF shall be greater than ST. Wherefore because the two Sides DE, EF, are equal to the two Sides SX, XT, and the Base DF is greater than the Base ST, the Angle DEF shall be greater than the Angle SXT; but the Angle SXT is equal to the Angles ABC, GHK. Therefore the Angle DEF, is greater than the Angles ABC, GHK; but it is also less, which is absurd; *which was to be demonstrated.*

PROPOSITION XXIV.

THEOREM.

If a Solid be contained under six parallel Planes, the opposite Planes thereof, are equal Parallelograms.

LET the Solid CDGH be contained under Parallel Planes AC, GF, BG, CE, FB, AE. I say, the opposite Planes thereof are equal Parallelograms.

For because the parallel Planes BG, CE, are cut by the Plane AC, their common Sections are * parallel; wherefore AB is parallel to CD. Again, because the two parallel Planes BF, AE, are cut by the Plane AC, their common Sections are parallel; therefore AD is parallel to BC; but AB has been proved to be parallel to CD; wherefore AC shall be a Parallelogram. After the same Manner, we demonstrate that CE, FG, GB, BF, or AE, is a Parallelogram.

* 16 *of this.*

Let AH, DF, be joined. Then because AB is parallel to DC, and BH to CF, the Lines AB, BH, touching each other, shall be parallel to the Lines DC, CF, touching each other, and not being in the same Plane; wherefore they shall † contain equal Angles. And so the Angle ABH is equal to the Angle DCF;

† 10 *of this.*

DCF. And since the two Sides AB, BH, are ‡ equal to the two Sides DC, CF, and the Angle ABH equal to the Angle DCF, the Base AH shall be * equal to the Base DF, and the Triangle ABH equal to the Triangle DCF. And since the Parallelogram BG is † double to the Triangle ABH, and the Parallelogram CE, to the Triangle DCF, the Parallelogram BG shall be equal to the Parallelogram CE. In like Manner, we demonstrate that the Parallelogram AC is equal to the Parallelogram GF, and the Parallelogram AE equal to the Parallelogram BF. *If*, therefore, *a Solid be contained under six parallel Planes, the opposites Planes thereof are equal Parallelograms*; which was to be demonstrated.

‡ 34. 1.
* 4. 1.
† 41. 1.

Coroll. It follows from what has been now demonstrated, that if a Solid be contained under six parallel Planes, the opposite Planes thereof are similar and equal, because each of the Angles are equal, and the Sides about the equal Angles are proportional.

PROPOSITION XXV.

THEOREM.

If a solid Parallelepipedon be cut by a Plane, parallel to opposite Planes; then as Base is to Base, so shall Solid be to Solid.

LET the solid Parallelepipedon ABCD, be cut by a Plane YE, parallel to the opposite Planes RA, DH. I say as the Base EFΦA is to the Base EHCF, so is the Solid ABFY to the Solid EGCD.

For let AH be both Ways produced, and make HM, MN, &c. equal to EH, and AK, KL, &c. equal to AE; and let the Parallelograms LO, KΦ, HX, MS, as likewise the Solids LP, KR, HΩ, MT, be compleated. Then because the Right Lines LK, KA, AE, are equal, the Parallelograms LO, KΦ, AF, shall be * also equal; as likewise the Parallelograms KΞ, KB, AG: And moreover † the Parallelograms LΨ, KP, AR, for they are opposite. For the same Reason, the Parallelograms EC, HX, MS, also are equal to each other; as also the Parallelograms

* 1. 6.
† 24 of this.

lelograms HG, HI, IN; and so are the Parallelograms DH, MΩ, NT. Therefore three Planes of the Solid LP, are equal to three Planes of the Solid KR, or AY, each to each; and the Planes opposite to these, are equal to them. Therefore the three Solids

‡ *Def.* 10 *of this.* LP, KR, AY, will be equal ‡ to each other. For the same Reason, the three Solids ED, HΩ, MT, are equal to each other. Therefore the Base LF is the same Multiple of the Base AF, as the Solid LY is of the Solid AY. For the same Reason, the Base NF is the same Multiple of the Base HF, as the Solid NY is of the Solid ED: And if the Base LF be equal to the Base NF, the Solid LY shall be equal to the Solid NY; and if the Base LF exceeds the Base NF, the Solid LY shall exceed the Solid NY; and if it be less, less. Wherefore because there are four Magnitudes, *viz.* the two Bases AF, FH, and the two Solids AY, ED, whose Equimultiples are taken, to wit, the Base LF, and the Solid LY; and the Base NF, and the Solid NY: And since is is proved, if the Base LF exceeds the Base NF, then the Solid LT will exceed the Solid NY, if equal, equal, and less, less. Therefore as the Base AF is

* *Def.* 6, 5. to the Base FH, so is * the Solid AY to the Solid ED. Wherefore, *if a solid Parallelepipedon be cut by a Plane, parallel to opposite Planes; then as Base is to Base, so shall Solid be to Solid*; which was to be demonstrated.

PROPOSITION XXVI.

THEOREM.

At a Right Line given, and at a Point given in it, to make a solid Angle equal to a solid Angle given.

LET AB be a Right Line given; A a given Point in it, and D a given solid Angle contained under the Plane Angles EDC, EDF, FDC; it is required to make a solid Angle at the given Point A, in the given Right Line AB, equal to the given solid Angle D.

Assume any Point F in the Right Line DF, from

* 11 *of this.* which let FG be drawn * perpendicular to the Plane passing

passing thro' ED, DC, meeting the said Plane in the Point G, and join DG, and make † the Angles BAL, BAK, at the given Point A, with the Right Line AB, equal to the Angles EDC, EDG. † 23. 1.

Lastly, make AK equal to DG, and at the Point K erect ‡ HK at Right Angles to the Plane passing thro' BAL, and make KH equal to GF, and join HA. I say, the solid Angle at A, which is contained under the three plane Angles BAL, BAH, HAL, is equal to the solid Angle at D, which is contained under the plane Angles EDC, EDF, FDC: For let the equal Right Lines AB, DE, be taken, and join HB, KB, FE, GE. Then because FG is perpendicular to the Plane passing thro' ED, DC, it shall be * perpendicular to all the Right Lines touching it that are in the said Plane. Wherefore both the Angles FGD, FGE, are Right Angles. For the same Reason, both the Angles HKA, HKB, are Right Angles; and because the two Sides KA, AB, are equal to the two Sides GD, DE, each to each, and contain equal Angles, the Base BK shall be † equal to the Base EG; but KH is also equal to GF, and they contain Right Angles; therefore HB shall be † equal to FE. Again, because the two Sides AK, KH, are equal to the two Sides DG, GF, and they contain Right Angles, the Base AH shall be equal to the Base DF; but AB is equal to DE. Therefore the two Sides HA, AB, are equal to the two Sides FD, DE; but the Base HB is equal to the Base FE, and so the Angle BAH will be ‡ equal to the Angle EDF. For the same Reason, the Angle HAL is equal to the Angle FDC; for since if AL be taken equal to DC, and KL, HL, GC, FC, be joined, the whole Angle BAL is equal to the whole Angle EDC; and the Angle BAK, a part of the one, is put equal to the Angle EDG, a part of the other; the Angle KAL remaining, will be equal to the Angle GDC remaining. And because the two Sides KA, AL, are equal to the two Sides GD, DC, and they contain equal Angles, the Base KL will be equal to the Base GC; but KH is equal to GF; wherefore the two Sides LK, KH, are equal to the two Sides CG, GF, but they contain Right Angles; therefore the Base HL will be equal to the Base FC,

‡ 12 *of this.*

* *Def.* 3. *of this.*

† 4. 1.

‡ 8. 1.

Again,

Again, because the two Sides H A, A L, are equal to the two Sides F D, D C, and the Base H L is equal to the Base F C, the Angle H A L will be equal to the Angle F D C; but the Angle B A L is equal to the Angle E D C; *which was to be done.*

PROPOSITION XXVII.

PROBLEM.

Upon a Right Line given, to describe a Parallelepipedon similar, and in like Manner situate to a solid Parallelepipedon given.

LET AB be a Right Line, and CD a given solid Parallelepipedon. It is required to describe a solid Parallelepipedon upon the given Right Line AB similar, and alike situate to the given solid Parallelepipedon CD.

Make a solid Angle at the given Point A, in the Right Line AB, which * is contained under the Angles BAH, HAK, KAB; so that the Angle BAH be equal to the Angle ECF, the Angle BAK to the Angle ECG, and the Angle HAK to the Angle GCF; and make as EC is to CG, so BA † to AK; and GC to CF, as KA to AH. Then, (by Equality of Proportion) as EC is to CF, so shall BA be to AH; compleat the Parallelogram BH, and the Solid AL. Then because it is as EC is to CG, so is BA to AK, *viz.* the Sides about the equal Angles ECG, BAK, proportional; the Parallelogram KB, shall be similar to the Parollelogram GE. Also, for the same Reason, the Parallelogram KH, shall be similar to the Parallelogram GF, and the Parallelogram HB, to the Parallelogram FE. Therefore three Parallelograms of the solid AL, are similar to three Parallelograms of the Solid CD; but these three Parallelograms are ‡ equal and similar to their three opposite ones. Therefore the whole Solid AL, will be similar to the whole Solid CD; and so a solid Parallelepipedon AL, is described upon the given Right

*26 *of this.*

‡ 12. 6.

‡ *Cor.* 24 *of this.*

Right Line AB similar, and alike situate to the given solid Parallelepipedon CD; *which was to be done.*

PROPOSITION XXVIII.

THEOREM.

If a solid Parallelepipedon be cut by a Plane passing thro' the Diagonals of two opposite Planes, that Solid will be bisected by the Plane.

LET the solid Parallelepipedon AB, be cut by the Plane CDEF, passing thro' the Diagonals CF, DE, of two opposite Planes. I say, the Solid AB is bisected by the Plane CDEF.

For because the Triangle CGF is * equal to the Triangle CBF, and the Triangle ADE to the Triangle DEH, and the Parallelogram CA to † the Parallelogram BE, for it is opposite to it; and the Parallelogram GE to the Parallelogram CH; the Prism contained by the two Triangles CGF, ADE, and the three Parallelograms GE, AC, CE, is equal to the Prism contained under the two Triangles CFB, DEH, and the three Parallelograms CH, BE, CE; for they are contained under Planes equal in Number and Magnitude. Therefore the whole Solid AB is bisected by the Plane CDEF; *which was to be demonstrated.*

* 34. 1.

† 24 *of this.*

PROPOSITION XXIX.

THEOREM.

Solid Parallelepipedons, being constituted upon the same Base, and having the same Altitude, and whose insistent Lines are in the same Right Lines, are equal to one another.

LET the solid Parallelepipedons CM, CN, be constituted upon the same Base AB, with the same Altitude, whose insistent Lines AF, AG, LM, LN, CD, CE, BH, BK, are in the same Right Lines FN, DK. I say, the Solid CM is equal to the Solid CN.

For

For because CH, CK, are both Parallelograms, CB shall be * equal to DH, or EK; wherefore DH is equal to EK. Let EH, which is common, be taken away, then the Remainder DE will be equal to the Remainder HK; and so the Triangle DEC is † equal to the Triangle HKB, and the Parallelogram DG equal to the Parallelogram HN. For the same Reason, the Triangle AFG is equal to the Triangle LMN; and the Parallelogram CF ‡ to the Parallelogram BM, and the Parallelogram CG to the Parallelogram BN, for they are opposite. Therefore the Prism contained under the two Triangles AFG, DEC, and the three Parallelograms AD, DG, GC, is equal to the Prism contained under the two Triangles LMN, HBK, and the three Parallelograms BM, NH, BN. Let the common Solid, whose Base is the Parallelogram AB, opposite to the Parallelogram GEHM, be added; then the whole solid Parallelepipedon CM, is equal to the whole solid Parallelepipedon CN. Therefore, *solid Parallelepipedons, being constituted upon the same Base, and having the same Altitude, and whose insistent Lines are in the same Right Lines, are equal to one another*; which was to be demonstrated.

* 34. 1.
† 8. 1.
‡ 24 of this.

PROPOSITION XXX.

THEOREM.

Solid Parallelepipedons, being constituted upon the same Base, and having the same Altitude, whose insistent Lines are not placed in the same Right Lines, are equal to one another.

LET there be solid Parallelepipedons CM, CN, having equal Altitudes, and standing on the same Base AB, and whose insistent Lines AF, AG, LM, LN, CD, CE, BH, BK, are not in the same Right Lines. I say, the Solid CM is equal to the Solid CN.

For let NK, DH, and GE, FM, be produced, meeting each other in the Points R, X; let also FM, GE, be produced to the Points O, P, and join AX, LO, CP, BR. The Solid CM, whose Base is the

Paral-

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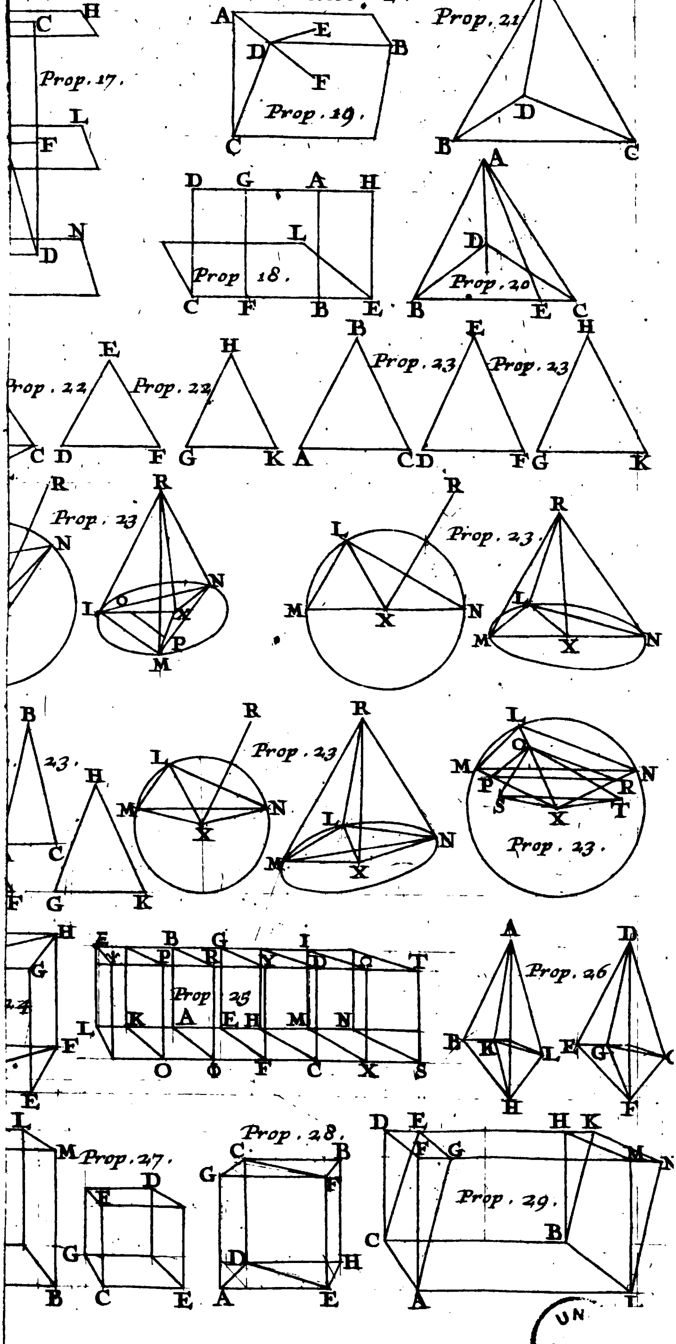


Parallelogram ACBL, being opposite to the Parallelogram FDHM, is * equal to the Solid CO, whose Base is the Parallelogram ACBL, being opposite to XPRO, for they stand upon the same Base ACBL; and the insistent Lines AF, AX, LM, LO, CD, CP, BH, BR, are in the same Right Lines FO, DR; but the Solid CO, whose Base is the Parallelogram ACBL, being opposite to XPRO, is * equal to the Solid CN, whose Base is the Parallelogram ACBL, being opposite to GEKN; for they stand upon the same Base ACBL, and their insistent Lines AG, AX, CE, CP, LN, LO, BK, BR, are in the same Right Lines GP, NR; wherefore the Solid CM shall be equal to the Solid CN. Therefore, *solid Parallelepipedons, being constituted upon the same Base, and having the same Altitude, whose insistent Lines are not placed in the same Right Lines, are equal to one another*; which was to be demonstrated. * 29 of this.

PROPOSITION. XXXI.

THEOREM.

Solid Parallelepipedons, being constituted upon equal Bases, and having the same Altitude, are equal to one another.

LET AE, CF, be solid Parallelepipedons constituted upon the equal Bases AB, CD, and having the same Altitude. I say, the Solid AE is equal to the Solid CF.

First, let HK, BE, AG, LM, OP, DF, CZ, RS, be at Right Angles to the Bases AB, CD; let the Angle ALB not be equal to the Angle CRD, and produce CR to T, so that RT be equal to AL: Then make the Angle TRY, at the Point R, in the Right Line RT, equal * to the Angle ALB; make RY equal to LB; draw XY thro' the Point Y parallel to RT, and compleat the Parallelogram RX, and the Solid ΦY. Therefore because the two Sides TR, RY, are equal to the two Sides AL, LB, and they contain equal Angles, the Parallelogram RX shall be equal and similar to the Parallelogram HL. * 23. 1.

And

And again, because AL is equal to RT, and LM to RS, and they contain equal Angles, the Parallelogram RΨ shall be equal and similar to the Parallelogram AM. For the same Reason, the Parallelogram LE is equal and similar to the Parallelogram SY. Therefore three Parallelograms of the Solid AE, are equal and similar to three Parallelograms of the Solid ΦY; and so the three opposite ones of one Solid, are † also equal and similar to the three opposite ones of the other. Therefore the whole solid Parallelepipedon AE is equal to the whole solid Parallelepipedon ΦY. Let DR, XY, be produced, and meet each other in the Point Ω, and let TQ be drawn thro' T parallel to DΩ, and produce TQ, OD, till they meet in V, and compleat the Solid ΩΨRI: Then the Solid ΨΩ, whose Base is the Parallelogram RΨ, and ΩΓ is that opposite to it, is ‡ equal to the Solid ΨY, whose Base is the Parallelogram RΨ, and YΦ is that opposite to it; for they stand upon the same Base RΨ, have the same Altitude, and their insistent Lines RΩ, RY, TQ, TX, SZ, SN, ΨΓ, ΨΦ, are in the same Right Lines ΩX, ZΦ: But the Solid ΨY is equal to the Solid AE; and so AE is equal to the Solid ΨΩ. Again, because the Parallelogram RYXT is equal to the Parallelogram ΩT, for it stands on the same Base RT, and between the same Parallels RT, ΩX; and the Parallelogram RYXT is equal to the Parallelogram CD, because it is also equal to AB; the Parallelogram ΩT is equal to the Parallelogram CD, and DT is some other Parallelogram. Therefore as the Base CD is to the Base DT, so is ΩT to TD, and because the solid Parallelepipedon CI is cut by the Plane RF, being parallel to two opposite Planes, it shall be * as the Base CD is to the Base DT, so is the Solid CF to the Solid RI. For the same Reason, because the solid Parallelepipedon ΩI is cut by the Plane RΨ parallel to two opposite Planes; as the Base ΩT is to the Base DT, so shall * the Solid ΩΨ be to the Solid RI; but as the Base CD is to the Base DT, so is the Base ΩT to TD. Therefore as the Solid CF is to the Solid RI, so is the Solid ΩΨ to the Solid RI; and since each of the Solids CF, ΩΨ, has the same Proportion to the Solid RI, the Solid

† 24 of this.

‡ 29 of this.

* 25 of this.

Solid CF is equal to the Solid ΩΨ; but the Solid ΩΨ has been proved equal to the Solid AE; therefore the Solid AE shall be † equal to the Solid CF. † 9. 5.

But now let the insistent Lines AG, HK, BE, LM, CN, OP, DF, RS, not be at Right Angles to the Bases AB, CD. I say, again, that the Solid AE is equal to the Solid CF. Let there be drawn from the Points K, E, G, M, P, F, N, S, to the Plane wherein are the Bases AB, CD, the Perpendiculars KΞ, ET, GY, MΦ, SI, FΨ, NΩ, PX, meeting the Plane in the Points Ξ, T, Y, Φ, I, Ψ, Ω, X, and join ΞT, YΦ, ΞY, TΦ, XΨ, XΩ, ΩI, ΨI; then the Solid KΦ is equal to the Solid PI, for they stand on equal Bases KM, PS, have the same Altitude, and the insistent Lines are at Right Angles to the Bases; but the Solid KΦ is equal to the Solid AE, and the Solid PI to ‡ the Solid CF, since they stand upon the same Base, have the same Altitude, and their insistent Lines are in the same Right Lines. Therefore the Solid AE shall be equal to the Solid CF. Wherefore, *solid Parallelepipedons, being constituted upon equal Bases, and having the same Altitude, are equal to one another*; which was to be demonstrated. ‡ 29 of this.

PROPOSITION XXXII.

THEOREM.

Solid Parallelepipedons, that have the same Altitude, are to each other as their Bases.

LET AB, CD, be solid Parallelepipedons that have the same Altitude. I say they are to one another as their Bases; that is, as the Base AE is to the Base CF, so is the Solid AB to the Solid CD.

For apply a Parallelogram FH to the Right Line FG, equal to the Parallelogram AE, and compleat the solid Parallelepipedon GK upon the Base FH, having the same Altitude as CD has. Then the solid AB is * equal to the solid GK; for they stand upon equal Bases AE, EH, and have the same Altitude; and so because the solid Parallelepipedon CK is cut by the Plane DG, parallel to two opposite Planes, it shall be † as the Base HF, is to the Base FC, * 31 of this. † 25 of this.

F C, so is the Solid HD to the Solid DC; but the Base FH is equal to the Base AE, and the Solid AB to the Solid FK. Therefore as the Base AE is to the Base CF, so is the Solid AB to the Solid CD. Wherefore *solid Parallelepipedons that have the same Altitude, are to each other as their Bases*; which was to be demonstrated.

PROPOSITION XXXIII.

THEOREM.

Similar solid Parallelepipedons, are to one another in the triplicate Proportion of their homologous Sides.

LET AB, CD, be solid Parallelepipedons, and let the Side AE be homologous to the Side CF. I say, the Solid AB, to the Solid CD, hath a Proportion triplicate of that which the Side AE has to the Side CF.

For produce AE, GE, HE, EK, EL, EM; and make EK equal to CF, and EL to FN, and EM to FR; and let the Parallelogram KL, and likewise the Solid KO be compleated. Then because the two Sides KE, EL, are equal to the two Sides CF, FN, and the Angle KEL equal to the Angle CFN; (since the Angle AEG is also equal to the Angle CFN, because of the Similarity of the Solids AB, CD,) the Parallelogram KL shall be similar and equal to the Parallelogram CN. For the same Reason, the Parallelogram KM is equal and similar to the Parallelogram CR, and the Parallelogram OE to DF. Therefore three Parallelograms of the Solid KO, are equal and similar to three Parallelograms of the solid CD: But those three Parallelograms are * equal and similar to the three opposite Parallelograms. Therefore the whole Solid KO is equal and similar to the whole Solid CD. Let the Parallelogram GK be compleated, as also the Solids EX, LP, upon the Bases GK, KL, having the same Altitude as AB. And since, because of the Similarity of the Solids AB and CD, it is as AE is to CF, so is EG to FN; and so EH to FR, and FC is equal to EK, and FN to EL, and FR to EM. It shall be as AE is to EK,

* 24 *of this.*

EK, so is † the Parallelogram AG to the Parallelogram GK; but as GE is to EL, so is GK to KL; and as HE is † to EM, so is PE to KM. Therefore as the Parallelogram AG is to the Parallelogram GK, so is GK to KL, and PE to KM. But as AG is to GK, so is ‡ the Solid AB to the Solid EX; and as GK is to KL, so is the Solid EX to the Solid PL; and as PE is to KM, so is the Solid PL to the Solid KO. Therefore as the Solid AB is to the Solid EX, so is * EX to PL, and PL to KO. But if four Magnitudes be continually proportional, the first to the fourth hath † a triplicate Proportion of that which it has to the second. Therefore also the Solid AB to the Solid KO, hath a triplicate Proportion of that which AB has to EX: But as AB is to EX, so is the Parallelogram AG to the Parallelogram GK; and so is the Right Line AE to the Right Line EK. Wherefore the Solid AB, to the Solid KO, hath a Proportion triplicate of that which AE has to EK; but the Solid KO is equal to the Solid CD, and the Right Line EK equal to the Right Line CF. Therefore the Solid AB to the Solid CD, has a Proportion triplicate of that which the homologous Side AE has to the homologous Side CF; *which was to be demonstrated.*

† 1. 6.

‡ 32 *of this.*

* 11. 5.

† *Def.* 11. 5.

Coroll. From hence it is manifest, if four Right Lines be proportional, as the first is to the fourth, so is a solid Parallelepipedon described upon the first, to a similar Solid Parallelepipedon alike situate described upon the second; because the first to the fourth, has a Proportion triplicate of that which it has to the second.

PROPOSITION XXXIV.

THEOREM.

The Bases and Altitudes of equal solid Parallelepipedons, are reciprocally proportional; and those solid Parallelepipedons whose Bases and Altitudes are reciprocally proportional, are equal.

LET AB, CD, be equal solid Parallelepipedons. I say, their Bases and Altitudes are reciprocally proportional, that is, as the Base EH is to the Base NP, so is the Altitude of the Solid CD to the Altitude of the Solid AB.

First, let the insistent Lines AG, EF, LB, HK, CM, NX, OD, PR, be at Right Angles to their Bases. I say, as the Base EH is to the Base NP, so is CM to AG. For if the Base EH be equal to the Base NP, and the Solid AB is equal to the Solid CD, the Altitude CM shall also be equal to the Altitude AG: For if when the Bases EH, NP, are equal, the Altitudes AG, CM, are not so; then the Solid AB will not be equal to the Solid CD, but it is put equal to it. Therefore the Altitude CM is not unequal to the Altitude AG, and so they are necessarily equal to one another; and consequently, as the Base EH is to the Base NP, so shall CM be to AG. But now let the Base EH be unequal to the Base NP, and let EH be the greater: Then since the Solid AB is equal to the Solid CD, CM is greater than AG; for otherwise it would follow that the Solids AB, CD, are not equal, which are put such. Therefore make CT equal to AG, and compleat the solid Parallelepipedon VC upon the Base NP, having the Altitude CT. Then because the Solid AB is equal to the Solid CD, and VC is some other Solid,
* 7. 5. and since equal Magnitudes have * the same Proportion to the same Magnitude, it shall be as the Solid AB is to the Solid CV, so is the Solid CD to the Solid CV; but as the Solid AB is to the Solid CV,
† 32 of this. so is † the Base EH to the Base NP; for AB, CV, are Solids having equal Altitudes. And as the Solid
‡ 25 of this. CD is to the Solid CV, so is ‡ the Base MP to the

Base PT, and so is MC to CT. Therefore as the Base EH is to the Base NP, so is MC to CT; but CT is equal to AG. Wherefore as the Base EH is to the Base NP, so is MC to AG. Therefore the Bases and Altitudes of the equal solid Parallelepipedons AB, CD, are reciprocally proportional.

Now, let the Bases and Altitudes of the solid Parallelepipedons AB, CD, be reciprocally proportional; that is, let the Base EH be to the Base NP, as the Altitude of the Solid CD is to the Altitude of the Solid AB. I say the Solid AB is equal to the Solid CD.

For let again the insistent Lines be at Right Angles to the Bases; then if the Base EH be equal to the Base NP, and EH is to NP, as the Altitude of the Solid CD is to the Altitude of the Solid AB; the Altitude of the Solid CD shall be equal to the Altitude of the Solid AB. But solid Parallelepipedons that stand upon equal Bases, and have the same Altitude, are * equal to each other. Therefore the Solid AB is equal to the Solid CD. * 30 of this.

But now let the Base EH not be equal to the Base NP, and let EH be the greater; then the Altitude of the Solid CD is greater than the Altitude of the Solid AB; that is, CM is greater than AG. Again, put CT equal to AG, and compleat the Solid CV as before: And then because the Base EH is to the Base NP, as MC is to AG, and AG is equal to CT; it shall be as the Base EH is to the Base NP, so is MC to CT; but as the Base EH is to the Base NP, so is the Solid AB to the Solid VC; for the Solids AB, CV, have equal Altitudes: And as MC is to CT, so is the Base MP to the Base PT, and so the Solid CD to the Solid CV. Therefore as the Solid AB is to the Solid CV, so is the Solid CD to the Solid CV: But since each of the Solids AB, CD, has the same Proportion to CV, the Solid AB shall be equal to the Solid CD; *which was to be demonstrated.*

Now let the insistent Lines FE, BL, GA, KH, XN, DO, MC, RP, not be at Right Angles to the Bases; and from the Points F, G, B, K, X, M, D, R, let there be drawn Perpendiculars to the Planes of the Bases EH, NP, meeting the same in the

Points S, T, Y, V, Q, Z, Ω, Φ, and compleat the Solids FV, XΩ. Then, I ſay, if the Solids AB, CD, be equal, their Baſes and Altitudes are reciprocally proportional, *viz.* as the Baſe EH is to the Baſe NP, ſo is the Altitude of the Solid CD to the Altitude of the Solid AB.

For becauſe the Solid AB is equal to the Solid CD, and the Solid AB is * equal to the Solid BT; for they ſtand upon the ſame Baſe, have the ſame Altitude, and their inſiſtent Lines are not in the ſame Right Lines, and the Solid DC is * equal to the Solid DZ, ſince they ſtand upon the ſame Baſe, XR have the ſame Altitude, and their inſiſtent Lines are not in the ſame Right Lines; the Solid BT ſhall be equal to the Solid DZ; but the Baſes and Altitudes of thoſe equal Solids, whoſe Altitudes are at Right Angles to their Baſes, are ‡ reciprocally proportional. Therefore as the Baſe FK is to the Baſe XR, ſo is the Altitude of the Solid DZ, to the Altitude of the Solid BT; but the Baſe FK is equal to the Baſe EH, and the Baſe XR to the Baſe NP. Wherefore as the Baſe EH is to the Baſe NP, ſo is the Altitude of the Solid DZ to the Altitude of the Solid BT; but the Solids DZ, DC, have the ſame Altitude, and ſo have the Solids BI, BA. Therefore the Baſe EH is to the Baſe NP, as the Altitude of the Solid DC is to the Altitude of the Solid AB; and ſo the Baſes and Altitudes of equal Solids are reciprocally proportional.

* 30 *of this.*

‡ *From what has been before proved.*

Again, let the Baſes and Altitudes of the ſolid Parallelepipedons AB, CD, be reciprocally proportional, *viz.* as the Baſe EH is to the Baſe NP, ſo let the Altitude of the Solid CD be to the Altitude of the Solid AB. I ſay, the Solid AB is equal to the Solid CD.

For the ſame Conſtruction remaining, becauſe the Baſe EH is to the Baſe NP, as the Altitude of the Solid CD is to the Altitude of the Solid AB; and ſince the Baſe EH is equal to the Baſe FK, and NP to XR. It ſhall be as the Baſe FK is to the Baſe XR, ſo is the Altitude of the Solid CD to the Altitude of the Solid AB; but the Altitudes of the Solids AB, BT, are the ſame; as alſo of the Solids CD, DZ. Therefore the Baſe FK is to the Baſe XR, as the

Altitude of the Solid DZ is to the Altitude of the Solid BT; wherefore the Bases and Altitudes of the solid Parallelepipedons BT, DZ, are reciprocally proportional; but those solid Parallelepipedons, whose Altitudes are at Right Angles to their Bases, and the Bases and Altitudes are reciprocally proportional, are equal to each other. But the Solid BT is equal to the Solid BA; for they stand upon the same Base FK, have the same Altitude, and their insistent Lines are not in the same Right Lines; and the Solid DZ is also equal to the Solid DC, since they stand upon the same Base XR, have the same Altitude, and their insistent Lines are not in the same Right Lines. Therefore the Solid AB is equal to the Solid CD; *which was to be demonstrated.*

PROPOSITION XXXV.

THEOREM.

If there be two plane Angles equal, and from the Vertices of those Angles two Right Lines be elevated above the Planes, in which the Angles are, containing equal Angles with the Lines first given, each to its correspondent one; and if in those elevated Lines any Points be taken from which Lines be drawn perpendicular to the Planes in which the Angles first given are, and Right Lines be drawn to the Angles first given from the Points made by the Perpendiculars in the Planes, those Right Lines will contain equal Angles with the elevated Lines.

LET BAC, EDF, be two equal Right-lin'd Angles; and from A, D, the Vertices of those Angles, let two Right Lines AG, DM, be elevated above the Planes of the said Angles, making equal Angles with the Lines first given, each to its correspondent one, *viz.* the Angle MDE equal to the Angle GAB, and the Angle MDF to the Angle GAC; and take any Points G and M in the Right Lines AG, DM, from which let GL, MN, be drawn perpendicular to the Planes passing thro BAC, EDF, meeting the same in the Points L, N, and

join LA, ND. I say the Angle GAL is equal to the Angle MDN.

Make AH equal to DM, and thro' H let HK be drawn parallel to GL; but GL is perpendicular to the Plane passing thro' BAC. Therefore HK shall
* 8 of this. be * also perpendicular to the Plane passing thro' BAC. Draw from the Points K, N, to the Right Lines AB, AC, DE, DF, the Perpendiculars KB, KC, NE, NF, and join HC, CB, MF, FE. Then because
† 47. 1. the Square of HA is † equal to the Squares of HK, KA, and the Squares of KC, CA, are † equal to the Square of KA; the Square of HA shall be equal to the Squares of HK, KC, CA; but the Square of HC is equal to the Squares of HK, KC. Therefore the Square of HA will be equal to the Squares of
‡ 48. 1. HC and CA; and so the Angle HCA is ‡ a Right Angle. For the same Reason, the Angle DFM is also a Right Angle. Therefore the Angle ACH is equal to DFM; but the Angle HAC is also equal to the Angle MDF. Therefore the two Triangles MDF, HAC, have two Angles of the one equal to two Angles of the other, each to each, and one Side of the one equal to one Side of the other, *viz.* that which is subtended by one of the equal Angles; that is, the Side HA equal to DM; and so the other
* 26. 1. Sides of the one, shall be * equal to the other Sides of the other, each to each. Wherefore AC is equal to DF. In like Manner we demonstrate that AB is equal to DE; for let HB, ME, be joined. Then because the Square of AH is equal to the Squares of AK and KH; and the Squares of AB, BK, are equal to the Square of AK; the Squares of AB, BK, KH, will be equal to the Square of AH; but the Square of BH is equal to the Squares of BK, KH; for the Angle HKB is a Right Angle, because HK is perpendicular to the Plane passing thro' BAC. Therefore the Square of AH is equal to the Squares
† 48. 1. of AB, BH. Wherefore the Angle ABH is † a Right Angle. For the same Reason, the Angle DEM is also a Right Angle. And the Angle BAH is equal to the Angle EDM, for so it is put; and AH is equal to DM. Therefore AB is * also equal to DE. And so since AC is equal to DF, and AB to DE, the two Sides CA, AB, shall be equal to the

the two Sides FD, DE; but the Angle BAC is equal to the Angle FDE. Therefore the Base BC is * equal to the Base EF, the Triangle to the Triangle, and the other Angles to the other Angles. Wherefore the Angle ACB is equal to the Angle DFE; but the Right Angle ACK is equal to the Right Angle DFN; and therefore the remaining Angle BCK is equal to the remaining Angle EFN. For the same Reason, the Angle CBK is equal to the Angle FEN; and so because BCK, EFN, are two Triangles, having two Angles equal to two Angles, each to each, and one Side equal to one Side, which is at the equal Angles, *viz.* BC equal to EF; therefore they shall have the other Sides equal to the other Sides. Therefore CK is equal to FN, but AC is equal to DF. Therefore the two Sides AC, CK, are equal to the two Sides DF, FN, and they contain Right Angles; consequently the Base AK is equal to the Base DN. And since AH is equal to DM, the Square of AH shall be equal to the Square of DM; but the Squares of AK, KH, are equal to the Square of AH; for the Angle AKH is a Right Angle, and the Squares DN, NM, are equal to the Square of DM, since the Angle DNM is a Right Angle. Therefore the Squares of AK, KH, are equal to the Squares of DN, NM; of which the Square of AK is equal to the Square of DN. Wherefore the Square of KH remaining, is equal to the remaining Square of NM; and so the Right Line HK is equal to MN. And since the two Sides HA, AK, are equal to the two Sides MD, DN, each to each, and the Base HK has been proved equal to the Base NM, the Angle HAK shall be † equal to the Angle MDN; *which was to be demonstrated.*

* 4. 1.

† 8. 1.

Coroll. From hence it is manifest, that if there be two Right-lin'd plane Angles equal, from whose Points equal Right Lines be elevated on the Planes of the Angles, containing equal Angles with the Lines first given, each to each; Perpendiculars drawn from the extreme Points of those elevated Lines to the Planes of the Angles first given, are equal to one another.

PROPOSITION XXXVI.

THEOREM.

If three Right Lines be proportional, the Solid Parallelepipedon made of them, is equal to the solid Parallelepipedon made of the Middle Line, if it be an Equilateral one, and Equiangular to the aforesaid Parallelepipedon.

LET three Right Lines A, B, C, be proportional, *viz.* Let A be to B, as B is to C. I say, the Solid made of A, B, C, is equal to the equilateral Solid made of B, equiangular to that made on A, B, C.

Let E be a solid Angle contained under the three plane Angles DEG, GEF, FED; and make DE, GE, EF, each equal to B, and compleat the solid Parallelepipedon EK. Again, put LM equal to A,
* 26 of this. and at the Point L, at the Right Line LM, make * a solid Angle contained under the Plane Angles NLX, XLM, MLN, equal to the solid Angle E; and make LN equal to B, and LX to C. Then because A is to B, as B is to C, and A is equal to LM, and B to LN, EF, EG, or ED, and C to LX; it shall be as LM is to EF, so is GE to LX. And so the Sides about the equal Angles MLX, GEF, are reciprocally proportional. Wherefore the Parallelogram
† 14. 6. MX is † equal to the Parallelogram GF. And since the two plane Angles GEF, XLM, are equal, and the Right Lines LN, ED, being equal are erected at the angular Points containing equal Angles with the Lines first given, each to each; the Perpendiculars
‡ Cor. 35 of this. drawn ‡ from the Points N, D, to the Planes drawn thro' XLM, GEF, are equal one to another. Therefore the Solids LH, EK, have the same Altitude; but solid Parallelepipedons that have equal Bases, and the
* 31 of this. same Altitude, are * equal to each other. Therefore the Solid HL is equal to the Solid EK. But the Solid HL is that made of the three Right Lines A, B, C, and the Solid EK that made of the Right Line B. Therefore, *if three Right Lines be proportional, the solid Parallelepipedon made of them, is equal to the solid Parallelepipedon made of the Middle Line, if it be an*

equi-

equilateral one, and equiangular to the aforesaid Parallelepipedon; which was to be demonstrated.

PROPOSITION XXXVII.

THEOREM.

If four Right Lines be proportional, the solid Parallelepipedons similar, and in like manner described from them, shall be proportional. And if the solid Parallelepipedons, being similar, and alike described, be proportional, then the Right Lines they are described from, shall be proportional.

LET the four Right Lines AB, CD, EF, GH, be proportional, *viz.* let AB be to CD, as EF is to GH, and let the similar and alike situate Parallelepipedons KA, LC, ME, NG, be described from them. I say, KA is to LC, as ME is to NG.

For because the solid Parallelepipedon KA is similar to LC, therefore KA to LC shall have * a Proportion triplicate of that which AB has to CD. For the same Reason, the Solid ME to NG will have a triplicate Proportion of that which EF has to GH. But AB is to CD, as EF is to GH. Therefore AK is to LC, as ME is to NG. And if the Solid AK be to the Solid LC, as the Solid ME is to the Solid NG. I say, as the Right Line AB is to the Right Line CD, so is the Right Line EF to the Right Line GH. For because AK to LC has † a Proportion triplicate of that which AB has to CD, and ME to NG has a Proportion triplicate of that which EF has to GH, and since AK is to LC, as ME is to NG; it shall be as AB is to CD, so is EF to GH. Therefore, *if four Right Lines be proportional, the solid Parallelepipedons similar, and in like manner described from them, shall be proportional. And if the solid Parallelepipedons, being similar and alike described, be proportional, then the Right Lines they are described from, shall be proportional*; which was to be demonstrated.

* 33 *of this.*

† 33 *of this.*

PRO-

PROPOSITION XXXVIII.

THEOREM.

If a Plane be perpendicular to a Plane, and a Line be drawn from a Point in one of the Planes perpendicular to the other Plane, that Perpendicular shall fall in the common Section of the Planes.

LET the Plane CD be perpendicular to the Plane AB, let their common Section be AD, and let some Point E be taken in the Plane CD. I say, a Perpendicular, drawn from the Point E to the Plane AB, falls on AD.

For if it does not, let it fall without the same, as EF meeting the Plane AB in the Point F, and from the Point F let FG be drawn in the Plane AB perpendi-
* *Def.* 4 *of this.* cular to AD; this shall be * perpendicular to the Plane CD; and join EG. Then because FG is perpendicular to the Plane CD, and the Right Line EG in the Plane of CD touches it: The Angle FGE shall
† *Def.* 3. *of this.* be † a Right Angle. But EF is also at Right Angles to the Plane Angle AB; therefore the Angle EFG is a Right Angle. And so two Angles of the Triangle
‡ 17. 1. EFG, are equal to two Right Angles; which is ‡ absurd. Wherefore if a Right Line, drawn from the Point E perpendicular to the Plane AB, does not fall without the Right Line AD: And so it must necessarily fall on it. Therefore, *if a Plane be perpendicular to a Plane, and a Line be drawn from a Point in one of the Planes perpendicular to the other Plane, that Perpendicular shall fall in the common Section of the Planes*; which was to be demonstrated.

PRO-

PROPOSITION XXXIX.

THEOREM.

If the Sides of the opposite Planes of a solid Parallelepipedon be divided into two equal Parts, and Planes be drawn thro' their Sections; the common Section of the Planes, and the Diameter, the solid Parallelepipedon, shall divide eachother into two equal Parts.

LET the Sides of CF, AH, the opposite Planes of the solid Parallelepipedon AF, be cut in half in the Points K, L, M, N, X, O, P, R, and let the Planes KN, XR, be drawn thro' the the Sections: Also let YS be the common Section of the Planes, and DG the Diameter of the solid Parallelepipedon. I say, YS, DG, bisect each other, that is, YT is equal to TS, and DT to TG.

For join DY, YE, BS, SG. Then because DX is parallel to OE, the Alternate Angles DXY, YOE are * equal to one another. And because DX is equal OE, and YX to YO, and they contain equal Angles, the Base DY shall be † equal to the Base YE; and the Triangle DXY to the Triangle YOE, and the other Angles equal to the other Angles: Therefore the Angle XYD is equal to the Angle OYE; and so DYE is ‡ a Right Line. For the same Reason BSG is also a Right Line, and BS is equal to SG. Then because CA is equal and parallel to DB, as also to EG, DB shall be equal and parallel to EG; and the Right Lines DE, GB, join them: Therefore DE is * parallel to BG, and D, Y, G, S, are Points taken in each of them, and DG, YS, are joined. Therefore DG, YS, are † in one Plane. And since DE is parallel to BG; the Angle EDT shall be * equal to the Angle BGT, for they are Alternate. But the Angle DTY, is ‡ equal to the Angle GTS. Therefore DTY, GTS are two Triangles, having two Angles of the one equal to two Angles of the other, as likewise one Side of the one equal to one Side of the other, *viz.* the Side DY equal to the Side GS: For they are Halves of DE, BG: Therefore they shall have the other Sides of one, equal to the other Sides of

* 29. 1.
† 4. 1.
‡ 14. 1.
* 33. 1.
† 7 of this.
* 29. 1.
‡ 15. 1.

of the other; and so DT is equal to TG, and YT to TS. Wherefore, *if the Sides of the opposite Planes of a solid Parallelepepidon be divided into two equal Parts, and Planes be drawn thro' their Sections; the common Section of the Planes, and the Diameter of the solid Parallelepipedon, shall divide each other into two equal Parts*; which was to be demonstrated.

PROPOSITION XL.

THEOREM.

If of two triangular Prisms, one standing on a Base, which a Parallelogram, and the other on a Triangle, if their Altitudes from these Bases are equal, and the Parallelogram double to the Triangle; then those Prisms are equal to each other.

LET ABCDEF, GHKLMN be two Prisms of equal Altitude. The Base of one of which is the Parallelogram AF, and that of the other, the Triangle GHK, and let the Parallelogram AF be double to the Triangle GHK. I say the Prism AB CDEF is equal to the Prism GHKLMN.

For compleat the Solids AX, GO. Then because the Parallelogram AF is double to the Triangle GHK, and since the Parallelogram HK is * double to the Triangle GHK, the, Parallelogram AF shall be equal to the Parallelogram HK. But solid Parallelepipedons. that stand upon equal Bases, and have the same Altitude, are † equal to one another. Therefore the solid AX is equal to the Solid GO. But the Prism ABCDEF is half the Solid AX, and the Prism GHKLMN is half the Solid GO. Therefore the Prism ABCDEF is equal to the Prism GH KLMN. Wherefore, *if there be two Prisms having equal Altitudes, the Base of one of which is a Parallelogram, and that of the other a Triangle, and if the Parallelogram be double to the Triangle, the said Prisms shall be equal to each other.*

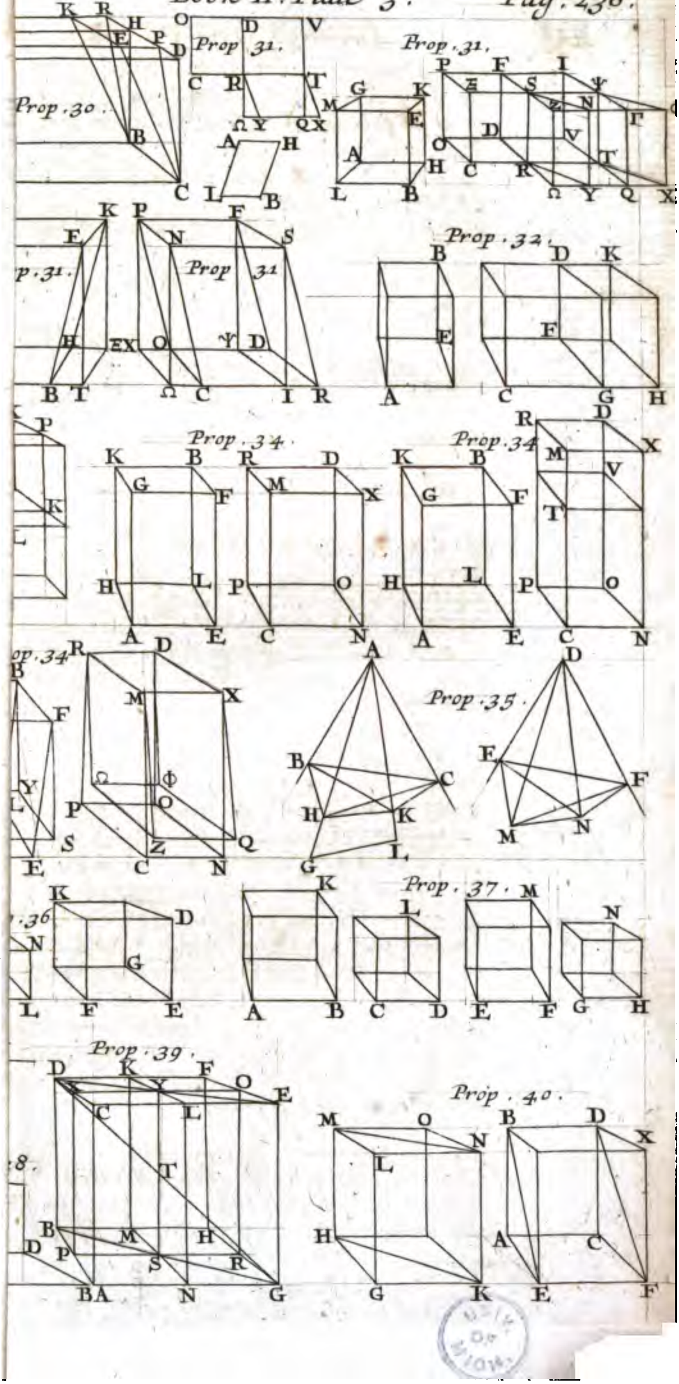
* 14. 1.

† 31 of this.

‡ 28 of this.

The END *of the* ELEVENTH BOOK.

EUCLID's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Prop. 31.
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Prop. 32.
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Prop. 35.
Prop. 37.
Prop. 39.
Prop. 40.

EUCLID's ELEMENTS.

BOOK XII.

PROPOSITION I.

THEOREM.

Similar Polygons, inscrib'd in Circles, are to one another as the Squares of the Diameters of the Circles.

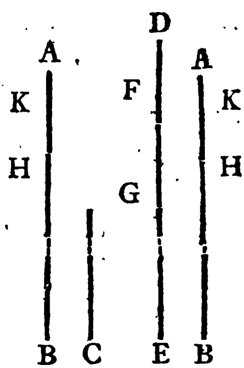
LET ABCDE, FGHKL, be Circles, wherein are inscrib'd the similar Polygons ABCDE, FGHKL, and let BM, GN, be Diameters of the Circles. I say, as the Square of BM is to the Square of GN, so is the Polygon ABCDE to the Polygon FGHKL.

For join BE, AM, GL, FN Then because the Polygon ABCDE is similar to the Polygon FGHKL, the Angle BAE is equal to the Angle GFL; and BA is to AE, as GF is to FL. Therefore the two Triangles BAE, GFL, have one Angle of the

the one equal to one Angle of the other, *viz.* the Angle BAE equal to the Angle GFL, and the Sides about the equal Angles proportional. Wherefore the
* 6. 6. Triangle ABE is * equiangular to the Triangle FGL; and so the Angle AEB is equal to the Angle FLG.
† 21. 3. But the Angle AEB is † equal to the Angle AMB; for they stand on the same Circumference; and the Angle FLG is † equal to the Angle FNG. Therefore the Angle AMB is equal to the Angle FNG.
‡ 31. 3. But the Right Angle BAM is ‡ equal to the Right Angle GFN. Wherefore the other Angle shall be equal to the other Angle. And so the Triangle AMB is equiangular to the Triangle FGN; and consequently
* 4. 6. * as BM is to GN, so is BA to GF. But the Proportion of the Square of BM to the Square of GN, is duplicate of the Proportion of BM to GN; and the Proportion of the Polygon ABCDE
† 22. 1. to the Polygon FGHKL, is † duplicate of the Proportion of BA to GF. Wherefore, as the Square of BM is to the Square of GN, so is the Polygon ABCDE to the Polygon FGHKL. Therefore, *similar Polygons, inscrib'd in Circles, are to one another as the Squares of the Diameters of the Circles*; which was to be demonstrated.

LEMMA.

If there be two unequal Magnitudes propos'd, and from the greater be taken a Part greater than its Half; and if from what remains there be again taken a Part greater than half this Remainder; and again from this last Remainder a Part greater than its half; and if this be done continually, there will remain at last a Magnitude that shall be less than the lesser of the propos'd Magnitudes.



LET AB and C be two unequal Magnitudes, whereof AB is the greater. I say, if from AB be

be taken a greater Part than half, and from the Part remaining there be again taken a Part greater than its half, and this be done continually, there will remain a Magnitude at last that shall be less than the Magnitude C.

For C being some number of Times multiply'd, will become greater than the Magnitude AB. Let it be multiply'd, and let DE be a Multiple of C greater than AB. Divide DE into Parts DF, FG, GE each equal to C, and take BH a Part greater than half of AB from AB, and again from AH the Part HK greater than half AH, and from AK a Part greater than half AK, and so on, until the Divisions that are in AB are equal in Number to the Divisions in DE. Therefore let the Divisions AK, KH, HB, be equal in Number to the Divisions DF, FG, GE. Then because DE is greater than AB, and the Part EG is taken from ED, being less than half thereof, and the Part BH greater than half of AB is taken from it, the Part remaining GD, shall be greater than the Part remaining HA. Again, because GD is greater than HA; and GF being half of GD, is taken from the same; and HK being greater than half HA, is taken from this likewise; the Part remaining FD, shall be greater than the Part remaining AK; but FD is equal to C. Therefore C is greater than AK; and so the Magnitude AK is lesser than C. Therefore the Magnitnde AK being the Part remaining of the Magnitude AB, is less than the lesser propos'd Magnitude C; *which was to be demonstrated.* If the Halves of the Magnitudes should have been taken, we demonstrate this after the same Manner. *This is the first Proposition of the tenth Book.*

PROPOSITION II.

THEOREM.

Circles are to each other as the Squares of their Diameters.

LET ABCD, EFGH, be Circles, whose Diameters are BD, FH. I say, as the Square of BD is to the Square of FH, so the Circle ABCD to the Circle EFGH. For

For if it be not so, the Square of BD shall be to the Square of FH, as the Circle ABCD is to some Space either less or greater than the Circle EFGH. First let it be to a Space S, less than the Circle EFGH, and let the Square EFGH be described therein.

This Square EFGH will be greater than half the Circle EFGH; because if we draw Tangents to the Circle thro' the Points E, F, G, H, the Square EFGH will be half that described about the Circle; but the Circle is less than the Square described about it. Therefore the Square EFGH is greater than half the Circle EFGH. Let the Circumferences EF, FG, GH, HE, be bisected in the Points K, L, M, N, and join EK, KF, FL, LG, GM, MH, HN, NE. Then each of the Triangles EKF, FLG, GMH, HNE, will be * greater than one half of the Segment of the Circle it stands in. Because if Tangents at the Circle be drawn thro' the Points K, L, M, N, and the Parallelograms that are on the Right Lines EF, FG, GH, HE be compleated, each of the Triangles EKF, FLG, GMH, HNE is half of each of the corresponding Parallelograms; but the Segment is less than the Parallelogram. Wherefore each of the Triangles EKF, FLG, GMH, HNE is greater than one half of the Segment of the Circle in which it stands. Therefore if these Circumferences be again bisected, and Right Lines be drawn joining the Points of Bisection, and you do thus continually, there will at last remain Segments of the Circle, that shall be less than the Excess, by which the Circle EFGH exceeds the Space S. For it is demonstrated in the aforegoing *Lemma*, that if there be two unequal Magnitudes proposed, and if from the greater a Part greater than half be taken from it, and again from the Part remaining a Part greater than half be taken, and you do this continually; there will at last remain a Magnitude that will be less than the lesser proposed Magnitude. Let the Segments of the Circle EFGH on the Right Lines EK, KF, FL, LG, GM, MH, HN, NE, be those which are less than the Excess, whereby the Circle EFGH exceeds the Space S, and then the remaining Polygon EKFLGMHN shall be greater than the Space S. Also describe the Polygon AXBOCPDR in the Circle ABCD, similar to the Polygon

* 41. 1.

lygon EKFLGMHN. Wherefore as the Square of BD is to the Square of FH so is the Polygon AXBOCPDR to * the Polygon EKFLGMHN. But as the Square of BD is to the Square of FH, so is the Circle ABCD to the Space S. Wherefore as the Circle ABCD to the Space S, so is † the Polygon AXBOCPDR to the Polygon EKFDGMHN. But the Circle ABCD is the greater than the Polygon in it. Wherefore the Space S shall be † also greater than the Polygon EKFLGMHN, but it is less ‡ likewise; which is absurd. Therefore the Square of BD to the Square of FH, is not as the Circle ABCD, to some Space less than the Circle EFGH. After the same Manner we likewise demonstrate that the Square of FH to the Square of BD is not as the Circle EFGH, to some Space less than the Circle ABCD. Lastly, I say, the Square of BD to the Square of FH is not as the Circle ABCD, to some Space greater than the Circle EFGH; for if it be possible, let it be so, and let the Space S be greater than the Circle EFGH; then shall it be (by Inversions) as the Square of FH is to the Square of BD, so is the Space S to the Circle ABCD. But because S is greater than the Circle EFGH, the Space shall be to the Circle ABCD, as the Circle EFGH is to some Space less than the Circle ABCD. Therefore, as the Square of FH is to the Square of BD, so is * the Circle EFGH to some Space less than the Circle ABCD; which has been demonstrated to be impossible. Wherefore the Square of BD to the Square of FH, is not as the Circle ABCD to some Space greater than the Circle EFGH. But this also has been proved, that the Square of BD to the Square of FH, is not as the Circle ABCD to some Space less than the Circle EFGH. Wherefore as the Square of BD is to the Square of FH, so shall the Circle ABCD be to the Circle EFGH. Wherefore Circles are to each other as the Squares of their Diameters; *which was to be demonstrated.*

* *1 of this.*
† *11. 5.*
‡ *from the Hyp.*
* *14. 5.*

PROPOSITION III.

THEOREM.

Every Pyramid having a triangular Base may be divided into two Pyramids, equal and similar to one another, having triangular Bases, and similar to the whole Pyramid, and into two equal Prisms, which two Prisms are greater than the half of the whole Pyramid.

LET there be a Pyramid, whose Base is the Triangle ABC; and Vertex the Point D. I say the Pyramid ABCD may be divided into Pyramids equal and similar to one another, having triangular Bases, and similar to the whole; and into two equal Prisms; which two Prisms are greater than the whole Pyramid.

For bisect AB, BC, CA, AD, DB, DC, in the Points E, F, G, H, K, L, and join EH, EG, GH, HK, KL, LH, EK, KF, FG. Then because AE
* 2. 6. is equal to EB, and AH to HD, EH shall be * parallel to DB. For the same Reason, HK also is parallel to AB. Therefore HEBK is a Parallelogram;
† 34. 1. and so HK is † equal to EB, but EB is equal to AE. Therefore AE shall be also equal to HK, but AH is equal to HD. Wherefore the two Sides AE, AH are equal to the two Sides KH, HD, each to each, and
‡ 29. 1. the Angle EAH is ‡ equal to the Angle KHD:
* 4. 1. Wherefore the Base EH is * equal to the Base KD: And so the Triangle AEH is equal and similar to the Triangle HKD. For the same Reason, the Triangle AHG shall also be equal and similar to the Triangle HLD. And because the two Right Lines EH, HG, touching each other, are parallel to the two Right Lines KD, DL, touching each other, and not in the same
† 10. 11. Plane with them, they shall contain † equal Angles. Therefore the Angle EHG is equal to the Angle KDL. Again, because the two Sides EH, HG, are equal to the two Sides KD, DL, each to each, and the Angle EHG is equal to the Angle KDL, the
* 4. 1. Base EG shall be * equal to the Base KL: And therefore, the Triangle EHG is equal and similar to the Triangle KDL. For the same Reason, the Triangle

angle

angle AEG is alſo equal and ſimilar to the Triangle HKL. Wherefore the Pyramid whoſe Baſe is the Triangle AEG, and Vertex the Point H, is equal and ſimilar to the Pyramid whoſe Baſe is the Triangle HKL, and Vertex the Point D. And becauſe HK is drawn parallel to the Side AB of the Triangle ADB, the Triangle ADB ſhall be equiangular to the Triangle DKH, and they have their Sides proportional. Therefore the Triangle ADB is ſimilar to the Triangle DHK. And for the ſame Reaſon, the Triangle DBC is ſimilar to the Triangle DKL; and the Triangle AHG to the Triangle DHL. And ſince the two Right Lines BA, AC, touching each other, are parallel to the two Lines KH, HL, touching each other, not being in the ſame Plane with them; theſe ſhall contain equal Angles. Therefore the Angle BAC is equal to the Angle KHL. And BA is to AC, as KH is to HL. Wherefore the Triangle ABC is ſimilar to the Triangle HKL; and ſo the Pyramid whoſe Baſe is the Triangle ABC, and Vertex the Point D, is ſimilar to the Pyramid, whoſe Baſe is the Triangle HKL, and Vertex the Point D. But the Pyramid whoſe Baſe is the Triangle HKL, and Vertex the Point D, has been proved ſimilar to the Pyramid whoſe Baſe is the Triangle AEG, and Vertex the Point H. Therefore the Pyramid whoſe Baſe is the Triangle ABC, and Vertex the Point D, is ſimilar to the Pyramid whoſe Baſe is the Triangle AEG, and Vertex the Point H. Wherefore both the Pyramids AEGH, HKLD, are ſimilar to the whole Pyramid ABCD. And becauſe BF is equal to FC, the Parallelogram EBFG will be double to the Triangle GFC. And ſince there are two Priſms of equal Altitude, one of which has that Parallelogram for a Baſe, and the other the Triangle, and the Parallelogram is double to the Triangle; thoſe Priſms will be † equal to one another. Therefore the Priſm contained under the two Triangles BKF, EHG, and the three Parallelograms EBFG, EBKH, KHGF, is equal to the Priſm contained under the two Triangles GFC, HKL, and the three Parallelograms KFCL, LCGH, HKFG. And it is Manifeſt that each of thoſe Priſms, the Baſe of one of which is the Parallelogram EBGF, and oppoſite Baſe to that the Right Line † 40. 11.

KH, and the Base of the other the Triangle GFC and the opposite Base to this, the Triangle KLH, are greater than either of the Pyramids, whose Bases are the Triangles AEG, HKL, and Vertices the Points H and D. For since, if the Right Lines EF, EH, be joined, the Prism whose Base is the Parallelogram EBFG, and opposite Base to that the Right Line KH, is greater than the Pyramid, whose Base is the Triangle EBF, and Vertex the Point K. But the Pyramid whose Base is the Triangle EBF, and Vertex the Point K, is equal to the Pyramid whose Base is the Triangle AEG, and Vertex the Point H. For they are contained under equal and similar Planes. Wherefore the Prism whose Base is the Parallelogram EBFG, and opposite Base to it the Right Line HK, is greater than the Ppramid whose Base is the Triangle AEG, and Vertex the Point H. But the Prism whose Base is the Parallelogram EBFG, and opposite Base to it the Right Line HK, is equal to the Prism whose Base is the Triangle GFC, and opposite Base to this the Triangle HKL: And the Pyramid whose Base is the Triangle AEG, and Vertex the Point H, is equal to the Pyramid whose Base is the Triangle HKL, and Vertex the Point D. Therefore the two Prisms aforesaid, are greater than the said two Pyramids, whose Bases are the Triangles AEG, HKL, and Vertices the Points H, D. And so the whole Pyramid whose Base is the Triangle ABC, and Vertex the Point D, is divided into two equal Pyramids, similar to each other and to the Whole: And into two equal Prisms; which two Prisms together are greater than half of the whole Pyramid. Therefore, *Every Pyramid having a triangular Base may be divided into two Pyramids, equal and similar to one another, having triangular Bases, and similar to the whole Pyramid, and into two equal Prisms, which two Prisms are greater than the half of the whole Pyramid*; which was to be demonstrated.

PRO-

PROPOSITION IV.

THEOREM.

If there are two Pyramids of the same Altitude, having triangular Bases, and each of them be divided into two Pyramids, equal to one another, and similar to the whole, as also into two equal Prisms; and if in like manner each of the two Pyramids, made by the former Division, be divided, and this be done continually; then as the Base of one Pyramid is to the Base of the other Pyramid, so are all the Prisms that are in one Pyramid to all the Prisms that are in the other Pyramid being equal in Multitude.

LET there be two Pyramids of the same Altitude, having the triangular Bases ABC, DEF, whose Vertices are the Points G, H, and let each of them be divided into two Pyramids, equal to one another, and similar to the whole, and into two equal Prisms; and if in like manner each of the Pyramids made by the former Division be conceived to be divided, and this be done continually. I say, as the Base ABC is to the Base DEF, so are all the Prisms that are in the Pyramid ABCG to all the Prisms that are in the Pyramid DEFH, being equal in Multitude.

For since BX is equal to XC, and AL to LC, XL shall be * parallel to AB, and the Triangle ABC * 2. 6.
similar to the Triangle LXC. For the same Reason the Triangle DEF shall be also similar to the Triangle RQF. And because BC is double to CX, and EF to FQ, it shall be as BC is to CX so is EF to FQ. And since there are describ'd upon BC, CX, Right-lin'd Figures ABC, LXC, similar and alike situate, and upon EF, FQ, Right-lin'd Figures DEF, RQF, similar and alike situate. Therefore as the Triangle BAC is to the Triangle LXC, so is † the † 22. 6.
Triangle DEF to the Triangle RQF; and (by Alternation) as the Triangle ABC is to the Triangle DEF, so is the Triangle LXC to the Triangle RQF. But as the Triangle LXC is to the Triangle RQF, so is ‡ the Prism, whose Base is the Triangle LXC, ‡ 28. &
and opposite Base to that the Triangle OMN, to the 32. 11.
Prism,

Prism whose Base is the Triangle RQF; and opposite Base to that the Triangle STY. Therefore as the
* 11. 5. Triangle ABC is to the Triangle DEF, so is * the Prism whose Base is the Triangle LXC, and opposite Base to that the Triangle OMN, to the Prism whose Base is the Triangle RQF, and opposite Base to that the Triangle STY; and because the two Prisms that are in the Pyramid ABCG are equal to one another, as also those two that are in the Pyramid DEFH; it shall be as the Prism whose Base is the Parallelogram KLXB, and opposite Base to that the Right Line MO, is to the Prism whose Base is the Triangle LXC; and opposite Base to that the Triangle OMN, so is the Prism whose Base is the Parallelogram EPRQ; and opposite Base to that the Right Line ST, to the Prism whose Base is the Triangle RQF, and opposite Base to that the Triangle STY. Therefore (by compounding) as the Prisms KBXLMO, LXCMNO, to the Prism LXCMNO, so the Prisms PEQRST, RQFSTY, to the Prism RQFSTY. And (by Alternation) as the Prisms KBXLMO, LXCMNO, to the Prisms PEQRST, RQFSTY, so the Prism LXCMNO, to the Prism RQFSTY; but as the Prism LXCMNO is to the Prism RQFSTY, so has the Base LXC been proved to be to the Base RFQ; and so the Base ABC to the Base DEF. Therefore also as the Triangle ABC is to the Triangle DEF, so are the two Prisms that are in the Pyramid ABCG, to the two Prisms that are in the Pyramid DEFH. If in the same Manner each of the Pyramids OMNG, STYH, made by the former Division, be divided, it shall be as the Base OMN is to the Base STY, so the two Prisms that are in the Pyramid OMNG, to the two Prisms that are in the Pyramid STYH. But as the Base OMN is to the Base STY, so is the Base ABC to the Base DEF. Therefore as the Base ABC is to the Base DEF, so the two Prisms that are in the Pyramid ABCG, to the two Prisms that are in the Pyramid DEFH; and so the two Prisms that are in the Pyramid OMNG, to the two Prisms that are in the Pyramid STYH, and so the four to the four. We demonstrate the same of Prisms made by the Division of the Pyramids AKLO, DPRS,

and

and of all other Prisms, being equal in Multitude; *which was to be demonstrated.*

PROPOSITION V.

THEOREM.

Pyramids of the same Altitude, and having triangular Bases, are to one another as their Bases.

LET there be two Pyramids of the same Altitude, having the triangular Bases ABC, DEF, whose Vertices are the Points G, H. I say, as the Base ABC, is to the Base DEF, so is the Pyramid AB CG to the Pyramid DEFH.

For if it be not so, then it shall be as the Base ABC is to the Base DEF, so is the Pyramid ABCG to some Solid, greater or less than the Pyramid DE FH. First, let it be to a Solid less, which let be Z, and divide the Pyramid DEFH into two Pyramids equal to each other, and similar to the Whole, and into two equal Prisms; then these two Prisms are greater than the half of the whole Pyramid. And again, let the Pyramids made by the former Division, be divided after the same Manner, and let this be done dontinually, until the Pyramids in the Pyramid DE FH, are less than the Excess by which the Pyramid DEFH exceeds the Solid Z. Let these, for Example, be the Pyramids DPRS, STYH; then the Prisms remaining in the Pyramid DEFH, are greater than the Solid Z. Also, let the Pyramid ABCG, be divided into the same Number of similar Parts as the Pyramid DEFH is; and then as the Base ABC is to the Base DEF, so * the Prisms that are in the Pyramid ABCG, to the Prisms that are in the Pyramid DEFH. But as the Base ABC is to the Base DEF, so is the Pyramid ABCG to the Solid Z. And therefore as the Pyramid ABCG is to the Solid Z, so are the Prisms that are in the Pyramid ABCG, to the Prisms that are in the Pyramid DEFH; but the Pyramid ABCG, is greater than the Prisms that are in it. Wherefore also the Solid Z, is greater than the Prisms that are in the Pyramid DEFH, but it

* 4 of this.

From what has been already demonstrated.

it is less * also, which is absurd. Therefore the Base ABC to the Base DEF, is not as the Pyramid ABCG to some Solid less than the Pyramid DEFH. After the same Manner we demonstrate that the Base DEF to the Base ABC, is not as the Pyramid DEFH to some Solid less than the Pyramid ABCG. Therefore, I say, neither is the Base ABC to the Base DEF, as the Pyramid ABCG to some Solid greater than the Pyramid DEFH. For if this is possible, let it be to the Solid I greater than the Pyramid DEFH. Then (by Inversion) the Base DEF shall be to the Base ABC, as the Solid I to the Pyramid ABCG: But since the Solid I is greater than the Pyramid EDFH, it shall be as the Solid I is to the Pyramid ABCG, so is the Pyramid DEFH to some Solid less than the Pyramid ABCG, as just now has been proved. And so as the Base DEF is to the Base ABC, so is the Pyramid DEFH, to some Solid less than the Pyramid ABCG, which is absurd. Therefore the Base ABC to the Base DEF, is not as the Pyramid ABCG to some Solid greater than the Pyramid DEFH. But it has been also proved, that the Base ABC to the Base DEF, is not as the Pyramid ABCG to some Solid less than the Pyramid DEFH. Wherefore as the Base ABC is to the Base DEF, so is the Pyramid ABCG to the Pyramid DEFH. Therefore, *Pyramids of the same Altitude, and having triangular Bases, are to one another as their Bases*; which was to be demonstrated.

PROPOSITION VI.

THEOREM.

Pyramids of the same Altitude, and having polygonous Bases, are to one another as their Bases.

LET there be Pyramids of the same Altitude, which have the polygonous Bases ABCDE, FGHKL, and let their Vertices be the Points M, N. I say, as the Base ABCDE is to the Base FGHKL, so is the Pyramid ABCDM to the Pyramid FGHKLN.

For

For let the Base ABCDE be divided into the Triangles ABC, ACD, ADE; and the Base FGHKL into the Triangles FGH, FHK, FKL; and let Pyramids be conceived upon every of those Triangles of the same Altitude with the Pyramids ABCDEM, FGHKLN. Then because the Triangle ABC is to the Triangle ACD, as * the Pyramid ABCM is to the Pyramid ACDM: And (by compounding) as the Trapezium ABCD is to the Triangle ACD, so is the Pyramid ABCDM to the Pyramid ACDM; but as the Triangle ACD is to the Triangle ADE, so is * the Pyramid ACDM to the Pyramid ADEM. Wherefore, (by Equality of Proportion) as the Base ABCD is to the Base ADE, so is the Pyramid ABCDM to the Pyramid ADEM. And again, (by Composition of Proportion) as the Base ABCDE is to the Base ADE, so is the Pyramid ABCDEM to the Pyramid ADEM. For the same Reason, as the Base FGHKL is to the Base FKL, so is the Pyramid FGHKLN to the Pyramid FKLN. And since there are two Pyramids ADEM, FKLN, having triangular Bases, and the same Altitude, the Base ADE shall be * to the Base FKL, as the Pyramid ADEM to the Pyramid FKLN. And since the Base ABCDE is to the Base ADE, as the Pyramid ABCDEM is to the Pyramid ADEM; and as the Base ADE is to the Base FKL, so is the Pyramid ADEM to the Pyramid FKLN; it shall be (by Equality of Proportion) as the Base ABCDE to the Base FKL, so is the Pyramid ABCDEM to the Pyramid FKLN; but as the Base FKL is to the Base FGHKL, so was the Pyramid FKLN to the Pyramid FGHKLN. Wherefore again, (by Equality of Proportion) as the Base ABCDE is to the Base FGHKL, so is the Pyramid ABCDEM to the Pyramid FGHKLN. Therefore, *Pyramids of the same Altitude, and having polygonous Bases, are to one another as their Bases*; which was to be demonstrated.

* 5 of this.

PRO-

PROPOSITION VII.

THEOREM.

Every Prism having a triangular Base, may be divided into three Pyramids equal to one another, and having triangular Bases.

LET there be a Prism whose Base is the Triangle ABC, and opposite Base to that the Triangle DEF. I say, the Prism ABCDEF may be divided into the three equal Pyramids that have triangular Bases.

For join BD, EC, CD. Then because ABED is a Parallelogram, whose Diameter is ED, the Triangle ABD shall be * equal to the Triangle EBD. Therefore the Pyramid whose Base is the Triangle ABD, and Vertex the Point C, is † equal to the Pyramid whose Base is the Triangle EDB, and Vertex the Point C. But the Pyramid whose Base is the Triangle EDB, and Vertex the Point C, is the same as the Pyramid whose Base is the Triangle EBC, and Vertex the Point D, for they are contained under the same Planes. Therefore the Pyramid, whose Base is the Triangle ABD, and Vertex the Point C, is equal to the Pyramid whose Base is the Triangle EBC, and vertex the Point D. Again, because FCBE is a Parallelogram, whose Diameter is CE, the Triangle ECF shall be * equal to the Triangle CBE. And so the Pyramid whose Base is the Triangle BEC, and Vertex the Point D, is † equal to the Pyramid whose Base is the Triangle ECF, and vertex the Point D: But the Pyramid, whose Base is the Triangle BCE, and Vertex the Point D, has been proved equal to the Pyramid, whose Base is the Triangle ABD, and Vertex the Point C. Wherefore also the Pyramid, whose Base is the Triangle CEF, and Vertex the Point D, is equal to the Pyramid, whose Base is the Triangle ABD, and vertex the Point C. Therefore the Prism ABCDEF is divided into three Pyramids equal to one another, and having triangular Bases. And because the Pyramid, whose Base is the Triangle ABD, and Vertex the Point C, is the same with the Pyramid whose Base is the Triangle CAB, and Vertex the Point D; for they

* 34. 1.

† 6. of this.

they are contained under the same Planes; and the Pyramid, whose Base is the Triangle ABD, and Vertex the Point C, has been proved to be a third Part of the Prism, whose Base is the Triangle ABC, and opposite Base to that the Triangle DEF. Therefore also the Pyramid, whose Base is the Triangle ABC, and Vertex the Point D, is a third Part of the Prism having the same Base, *viz.* the Triangle ABC, and the opposite Base the Triangle DEF; *which was to be demonstrated.*

Coroll. 1. It is manifest from hence, that every Pyramid is a third Part of a Prism, having the same Base and an equal Altitude; because if the Base of a Prism, as also the opposite Base, be of any other Figure, it may be divided into Prisms having triangular Bases.

2. Prisms of the same Altitude are to one another as their Bases.

PROPOSITION VIII.

THEOREM.

Similar Pyramids, having triangular Bases, are in a triplicate Proportion of their homologous Sides.

LET there be two Pyramids similar and alike situate, having the triangular Bases ABC, DEF, and let their Vertices be the Points G, H. I say, the Pyramid ABCG to the Pyramid DEFH, has a Proportion triplicate of that which BC has to EF.

For compleat the solid Parallelepipedons BGML, EHPO; then because the Pyramid ABCG is similar to the Pyramid DEFH, the Angle ABC shall be * equal to the Angle DEF, the Angle GBC equal to the Angle HEF, and the Angle ABG equal to the Angle DEH. And AB is to DE as BC is to EF; and so is BG to EH. Therefore because AB is to DE, as BC is to EF; and the Sides about the equal Angles are proportional, the Parallelogram BM shall † be similar to the Parallelogram EP. For the same Reason, the Parallelogram BN is similar to the Parallelogram ER, and the Parallelogram

* Def. 9. 11.

† 6. 6.

rallelogram BK to the Parallelogram EX. Therefore three Parallelograms BM, KB, BN, are similar to three Parallelograms EP, EX, ER; but the three M B, BK, BN, are equal and similar to the three opposite ones; as also the threee EP, EX, ER. Therefore the Solids BGML, EHPO, are contained under equal Numbers of similar and equal Planes; and consequently, the Solid BGML is similar to the Solid EHPO. But similar solid Parallelepipedons are * to each other in a triplicate Proportion of their homologous Sides. Therefore the Solid BGML to the Solid EHPO, has a Proportion triplicate of that which the homologous Side BC has to the homologous Side EF. But as the Solid BGML is to the Solid EHPO, so is † the Pyramid ABCG to the Pyramid DEFH; for the Pyramid is the one sixth Part of that Solid, since the Prism, which is the half of the solid Parallelepipedon is triple of the Pyramid. Wherefore the Pyramid ABCG to the Pyramid DEFH, shall have a triplicate Proportion to that which BC has to EF; *which was to be demonstrated.*

* 33. 11.

† 15. 5.

Coroll. From hence it is manifest, that similar Pyramids having polygonous Bases, are to one another in a triplicate Proportion of their homologous Sides. For if they be divided into Pyramids having triangular Bases: Because their similar polygonous Bases are divided into similar Triangles equal in Number, and homologous to the Wholes, it shall be as one Pyramid having a triangular Base in one of the Pyramids, is to a Pyramid having a triangular Base in the other Pyramid, so are all the Pyramids having triangular Bases in one Pyramid, to all the Pyramids having triangular Bases in the other Pyramid; that is, so is one of the Pyramids having the polygonous Base, to the other; but a Pyramid having a triangular Base to a Pyramid having a triangular Base, is in a triplicate Proportion of the homologous Sides. Therefore one Pyramid having a polygonous Base to another Pyramid having a similar Base, is in a triplicate Proportion of the homologous Sides.

PRO-

PROPOSITION IX.

THEOREM.

The Bases and Altitudes of equal Pyramids, having triangular Bases, are reciprocally proportional; and those Pyramids, having triangular Bases, whose Bases and Altitudes are reciprocally proportional, are equal.

LET there be equal Pyramids, having the triangular Bases ABC, DEF, and Vertices the Points G, H. I say the Bases and Altitudes of the Pyramids ABCG, DEFH, are reciprocally proportional, that is, as the Base ABC is to the Base DEF, so is the Altitude of the Pyramid DEFH to the Altitude of the Pyramid ABCG.

For compleat the solid Parallelepipedons BGML, EHPO. Then because the Pyramid ABCG is equal to the Pyramid DEFH, and the Solid BGML is sextuple, the Pyramid ABCG, and the Solid EHPO sextuple of the Solid DEFH, the Solid BGML shall be * equal to the Solid EHPO. But the Bases and Altitudes of equal solid Parallelepipedons are reciprocally proportional. Therefore, as the Base BM is to the Base EP, so is † the Altitude of the Solid EHPO to the Altitude of the Solid BGML. But as the Base BM is to the Base EP, so is † the Triangle ABC to the Triangle DEF. Therefore, as the Triangle ABC is to the Triangle DFF, so is the Altitude of the Solid EHPO to the Altitude of the Solid BGML. But the Altitude of the Solid EHPO is the same as the Altitude of the Pyramid DEFH; and the Altitude of the Solid BGML the same as the Altitude of the Pyramid ABCG. Therefore, as the Base ABC is to the Base DEF, so is the Altitude of the Pyramid DEFH to the Altitude of the Pyramid ABCG. Wherefore the Bases and Altitudes of the equal Pyramids ABCG, DEFH, are reciprocally proportional; and if the Bases and Altitudes of the Pyramids ABCG, DEFH, are reciprocally proportional, that is, if the Base ABC to the Base DEF, be as the Altitude of the Pyramid DEFH

* 15. 5.

† 34. 11.

DEFH

DEFH to the Altitude of the Pyramid ABCG. I say the Pyramid ABCG is equal to the Pyramid DEFH: For, the same Construction remaining, because the Base ABC to the Base DEF, is as the Altitude of the Pyramid DEFH to the Altitude of the Pyramid ABCG, and as the Base ABC is to the Base DEF, so is the Parallelogram BM to the Parallelogram EP; the Parallelogram BM to the Parallelogram EP shall be also as the Altitude of the Pyramid DEFH is to the Altitude of the Pyramid ABCG. But the Altitude of the Pyramid DEFH is the same as the Altitude of the solid Parallelepipedon EHPO, and the Altitude of the Pyramid ABCG the same as the Altitude of the solid Parallelepipedon BGML. Therefore the Base BM to the Base EP will be as the Altitude of the solid Parallelepipedon EHPO to the Altitude of the solid Parallelepipedon BGML. But those solid Parallelepipedons, whose Basis and Altitudes are reciprocally proportional,
† 34. 11. are † equal to each other. Therefore the solid Parallelepipedon BGML is equal to the solid Parallelepipedon EHPO; and the Pyramid ABCG is a sixth Part of the Solid BGML. And in like manner the Pyramid DEFH is a sixth Part of the Solid EHPO. Therefore the Pyramid ABCG is equal to the Pyramid DEFH. Wherefore *the Bases and Altitudes of equal Pyramids, having triangular Bases, are reciprocally proportional; and those Pyramids, having triangular Bases, whose Bases and Altitudes are reciprocally proportional, are equal*; which was to be demonstrated.

PROPOSITION X.

THEOREM.

Every Cone is a third Part of a Cylinder, having the same Base, and an equal Altitude.

LET a Cone have the same Base as a Cylinder, *viz.* the Circle ABCD, and an Altitude equal to it. I say the Cone is a third Part of the Cylinder, that is, the Cylinder is triple to the Cone.

For

For if the Cylinder be not triple to the Cone, it ſhall be greater or leſs than triple thereof. Firſt let it be greater than triple to the Cone, and let the Square ABCD be deſcrib'd in the Circle ABCD, then the Square ABCD is greater than one half of the Circle ABCD. Now let a Priſm be erected upon the Square ABCD, having the ſame Altitude as the Cylinder, and this Priſm will be greater than one half of the Cylinder; becauſe, if a Square be circumſcrib'd about the Circle ABCD, the inſcrib'd Square will be one half of the circumſcrib'd Square; and if a Priſm be erected upon the circumſcrib'd Square of the ſame Altitude as the Cylinder, ſince Priſms are * to one another as their Baſes, the Priſm erected upon the Square ABCD is one half of the Priſm erected upon the Square deſcrib'd about the Circle ABCD. But the Cylinder is leſſer than the Priſm erected on the Square deſcrib'd about the Circle ABCD. Therefore the Priſm erected on the Square ABCD, having the ſame Height as the Cylinder, is greater than one half of the Cylinder. Let the Circumferences AB, BC, CD, DA, be biſected in the Points E, F, G, H, and join AE, EB, BF, FC, CG, GD, DH, HA. Then each of the Triangles AEB, BFC, CGD, DHA, is † greater than the half of each of the Segments in which they ſtand. Let Priſms be erected from each of the Triangles AEB, BFC, CGD, DHA, of the ſame Altitude as the Cylinder, then every one of theſe Priſms erected is greater than its correſpondent Segment of the Cylinder. For becauſe, if Parallels be drawn thro' the Points E, F, G, H, to AB, BC, CD, DA, and Parallelograms be compleated on the ſaid AB, BC, CD, DA, on which are erected ſolid Parallelepipedons of the ſame Altitude as the Cylinder; then each of thoſe Priſms that are on the Triangles AEB, BFC, CGD, DHA, are Halves † of each of the ſolid Parallelepipedons; and the Segments of the Cylinder are leſs than the erected ſolid Parallelepipedons; and conſequently the Priſms that are on the Triangles AEB, BFC, CGD, DHA, are greater than the Halves of the Segments of the Cylinder; and ſo biſecting the other Circumferences, joining Right Lines, and on every of the Triangles erecting Priſms of the ſame Height as the Cylinder;

* 2 *Cor.* 7. *of this.*

† *This follows from* 2 *of this.*

Cylinder; and doing this continually, we shall at last have certain Portions of the Cylinder left, that are less than the Excess by which the Cylinder exceeds triple the Cone.

Now let these Portions remaining be AE, EB, BF, FC, CG, GD, DH, HA. Then the Prism remaining, whose Base is the Polygon AEBFCGDH, and Altitude equal to that of the Cylinders, is greater than the Triple of the Cone. But the Prism whose Base is the Polygon AEBFCGDH, and Altitude the same; as that of the Cylinders is * triple of the Pyramid whose Base is the Polygon AEBFCGDH, and Vertex the same as that of the Cone. And therefore the Pyramid whose Base is the Polygon AEBFCGDH, and Vertex the same as that of the Cone, is greater than the Cone whose Base is the Circle ABCD; but it is lesser also; (for it is comprehended by it) which is absurd. Therefore the Cylinder is not greater than triple the Cone. I say it is neither lesser than triple the Cone: For if it be possible, let the Cylinder be less than triple the Cone: Then (by Inversion) the Cone shall be greater than a third Part of the Cylinder. Let the Square ABCD be described in the Circle ABCD; then the Square ABCD is greater than half of the Circle ABCD. And let a Pyramid be erected on the Square ABCD having the same Vertex as the Cone, then the Pyramid erected is greater than one half of the Cone; because, as has been already demonstrated, if a Square be described about the Circle, the Square ABCD shall be half thereof. And if solid Parallelepipedons be erected upon the Squares of the same Altitude as the Cone, which are also called Prisms; then the Prism erected on the Square ABCD is one half of that erected on the Square described about the Circle, for they are to each other as their Bases; and so likewise are their third Parts. Therefore the Pyramid whose Base is the Square ABCD, is one half of that Pyramid erected upon the Square described about the Circle: But the Pyramid erected upon the Square described about the Circle, is greater than the Cone; for it comprehends it. Therefore the Pyramid whose Base is the Square ABCD, and Vertex the same as that of the Cone, is greater than one half of

* 1 Cor. 7 of this.

of the Cone. Bisect the Circumferences A B, BC, CD, D A, in the Points E, F, G, H, and join A E, EB, BF, FC, CG, GD, DH, HA; and then each of the Triangles AEB, BFC, CGD, DHA, is greater than one half of each of the Segments they are in. Let Pyramids be erected upon each of the Triangles AEB, BFC, CGD, DHA, having the same Vertex as the Cone; then each of these Pyramids thus erected, is greater than one half of the Segment of the Cone in which it is: And so bisecting the the remaining Circumferences, joining the Right Lines, and erecting Pyramids upon every of the Triangles having the same Altitude as the Cone; and doing this continually, we shall at last have Segments of the Cone left, that will be less than the Excess by which the Cone exceeds the one third Part of the Cylinder. Let these Segments be those that are on A E, EB, BF, FC, CG, GD, DH, HA, and then the remaining Pyramid whose Base is the Polygon AEBFCGDH, and Vertex the same as that of the Cone, is greater than a third Part of the Cylinder; but the Pyramid whose Base is the Polygon AEBFCGDH, and Vertex the same as that of the Cone, is one third Part of the Prism whose Base is the Polygon AEBFCGDH, and Altitude the same as that of the Cylinder. Therefore the Prism, whose Base is the Polygon AEBF CGDH, and Altitude the same as that of the Cylinder, is greater than the Cylinder whose Base is the Circle ABCD; but it is less also (as being comprehended thereby) which is absurd; therefore the Cylinder is not less than triple of the Cone; but it has been proved also not to be greater than triple of the Cone; therefore the Cylinder is necessarily triple of the Cone. Wherefore, *every Cone is a third Part of a Cylinder, having the same Base, and an equal Altitude*; which was to be demonstrated.

PROPOSITION XI.

THEOREM.

Cones and Cylinders of the same Altitude are to one another as their Bases.

LET there be Cones and Cylinders of the same Altitude, whose Bases are the Circles ABCD, EFGH, Axes KL, MN, and Diameters of the Bases AC, EG. I say, as the Circle ABCD is to the Circle EFGH, so is the Cone AL to the Cone EN.

For if it be not so, it shall be as the Circle ABCD is to the Circle EFGH, so is the Cone AL to some Solid either less or greater than the Cone EN. First, let it be to the Solid X less than the Cone; and let the Solid I be equal to the Excess of the Cone EN above the Solid X. Then the Cone EN is equal to the Solids X, I; let the Square EFGH be described in the Circle EFGH, which Square is greater than one half of the Circle, and erect a Pyramid upon the Square EFGH of the same Altitude as the Cone. Therefore the Pyramid erected is greater than one half of the Cone: For if we describe a Square about the Circle, and a Pyramid be erected thereon of the same Altitude as the Cone, the Pyramid inscribed will be one half of the Pyramid circumscribed, for they are
6 of this. * to one another as their Bases; and the Cone is less than the circumscribed Pyramid. Therefore the Pyramid whose Base is the Square EFGH, and Vertex the same as that of the Cone, is greater than one half of the Cone. Bisect the Circumferences EF, FG, GH, HE, in the Points P, R, S, O, and join HO, OE, EP, PF, FR, RG, GS, SH; then each of the Triangles HOE, EPF, FRG, GHS, is greater than one half of the Segment of the Circle wherein it is. Let a Pyramid be raised upon every one of the Triangles HOE, EPF, FRG, GHS, of the same Altitude as the Cone. Then each of those erected Pyramids is greater than the one half of its correspondent Segment of the Cone: And so bisecting the remaining Circumferences joining the Right Lines, and erecting Pyramids upon each of the Triangles of the

the ſame Altitude as that of the Cone; and doing this continually, there will at laſt be left Segments of the Cone that will together be leſs than the Solid I. Let thoſe be the Segments that are on HO, OE, EP, PE, FR, RG, GS, SH. Therefore the Pyramid remaining, whoſe Baſe is the Polygon HOEPFRGS, and Altitude the ſame as that of the Cone, is greater than the Solid X. Let the Polygon DTAYBQCV be deſcribed in the Circle ABCD, ſimilar and alike ſituate to the Polygon HOEPFRGS, and let a Pyramid be erected thereon of the ſame Altitude as the Cone AL. Then becauſe the Square of AC to the Square of EG, is * as the Polygon DTAYBQCV to the Polygon HOEPFRGS; and the Square of AC is † to the Square of EG, as the Circle ABCD to the Circle EFGH; it ſhall be as the Circle ABCD to the Circle EFGH, ſo is the Polygon DTAYBQCV to the Polygon HOEPFRGS: But as the Circle ABCD is to the Circle EFGH, ſo is the Cone AL to the Solid X; and as the Polygon DTAYBQCV is to the Polygon HOEPFRGS, ſo is the Pyramid whoſe Baſe is the Polygon DTAYBQCV, and Vertex the Point L, to the Pyramid whoſe Baſe is the Polygon HOEPFRGS, and Vertex the Point N. Therefore as the Cone AL to the Solid X, ſo the Pyramid whoſe Baſe is the Polygon DTAYBQCV, and Vertex the Point L, to the Pyramid whoſe Baſe is the Polygon HOEPFRGS, and Vertex the Point N; but the Cone AL is greater than the Pyramid that is in it. Therefore the Solid X is greater than the Pyramid that is in the Cone EN; but it was put leſs, which is abſurd. Therefore the Circle ABCD to the Circle EFGH, is not as the Cone AL to ſome Solid leſs than the Cone EN. In like Manner, it is demonſtrated that the Circle EFGH to the Circle ABCD, is not as the Cone EN to ſome Solid leſs than the Cone AL. I ſay, moreover, that that the Circle ABCD to the Circle EFGH, is not as the Cone AL to ſome Solid greater than the Cone EN: For, if it be poſſible, let it be to the Solid Z greater than the Cone; then, (by Inverſion) as the Circle EFGH is to the Circle ABCD, ſo ſhall the Solid Z be to the Cone AL. But ſince the Solid Z is greater than the Cone EN, it ſhall be as the Solid

* 1 of this.

† 2 of this.

Z is to the Cone AL, so is the Cone EN to some Solid less than the Cone AL. And therefore as the Circle EFGH is to the Circle ABCD, so is the Cone EN to some Solid less than the Cone AL; which has been proved to be impossible. Therefore the Circle ABCD to the Circle EFGH, is not as the Cone AL to some Solid greater than the Cone EN. It has also been proved that the Circle ABCD to the Circle EFGH, is not as the Cone AL to some Solid less than the Cone EN. Therefore as the Circle ABCD is to the Circle EFGH, so is the Cone AL to the Cone EN: But as Cone is to Cone, so
* 15. 5. is * Cylinder to Cylinder, for each Cylinder is triple of each Cone; and therefore as the Circle ABCD is to the Circle EFGH, so are Cylinders and Cones standing on them, of the same Altitude. Wherefore, *Cones and Cylinders of the same Altitude, are to one another as their Bases;* which was to be demonstrated.

PROPOSITION XII.

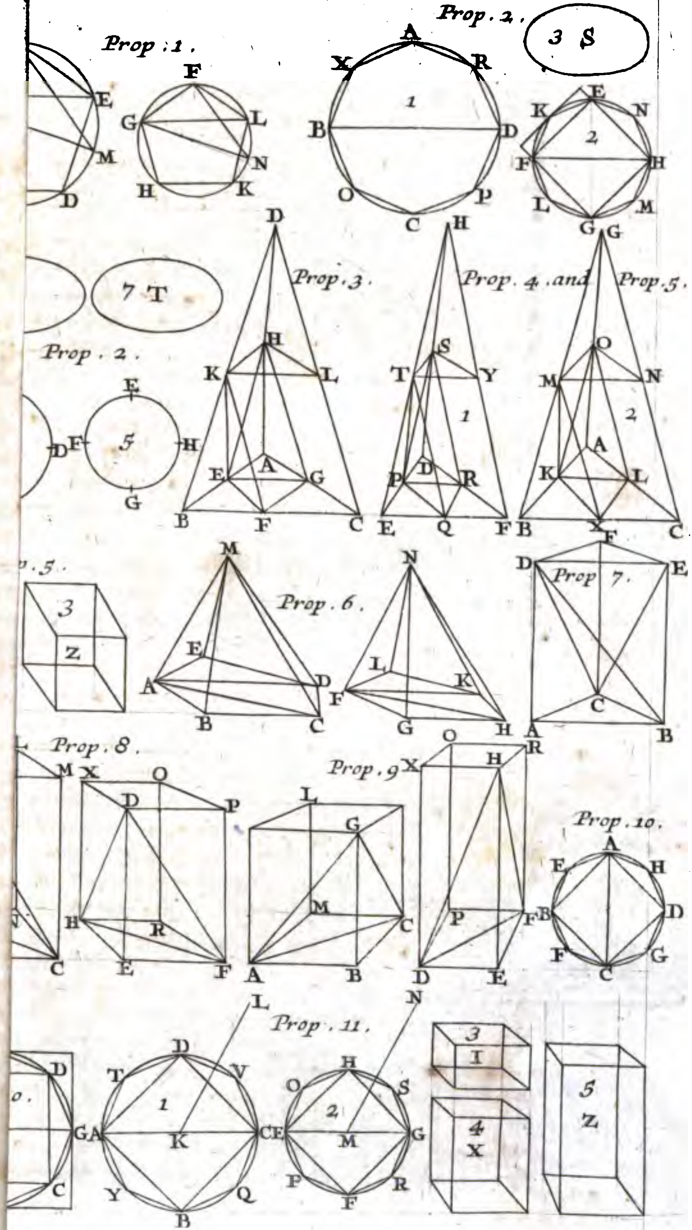
THEOREM.

Similar Cones and Cylinders are to one another in a triple Proportion of the Diameters of their Bases.

LET there be similar Cones and Cylinders, whose Bases are the Circles ABCD, EFGH, and Diameters of the Bases BD, FH, and Axes of the Cones or Cylinders KL, MN. I say, the Cone whose Base is the Circle ABCD, and Vertex the Point L, to the Cone whose Base is the Circle EFGH, and Vertex the Point N, hath a triplicate Proportion of that which BD has to FH.

For if the Cone ABCDL to the Cone EFGHN, has not a triplicate Proportion of that which BD has to FH. The Cone ABCDL shall have that triplicate Proportion to some Solid either less or greater than the Cone EFGHN. First, let it have that triplicate Proportion to the Solid X, less than the Cone EFGHN; and let the Square EFGH be described in the Circle EFGH, which will be greater than one half of the Circle EFGH; and erect a Pyramid on the

Book 12. Plate 1.



the Square EFGH of the same Altitude with the Cone, then that Pyramid is greater than one half of the Cone. And so let the Circumferences EF, FG, GH, HE, be bisected in the Points O, P, R, S, and join EO, OF, FP, PG, GR, RH, HS, SE; then each of the Triangles EOF, FPG, GRH, HSE, is greater than one half of the Segment of the Circle EFGH, in which it is; and erect a Pyramid upon each of the Triangles EOF, FPG, GRH, HSE, having the same Altitude as the Cone: Then each of the Pyramids thus erected, is greater than half its corresponding Segment of the Cone. Wherefore bisecting the remaining Circumferences joining Right Lines, and erecting Pyramids upon each of the Triangles, having the same Vertex as the Cone; and doing this continually, we shall leave at last certain Segments of the Cone that shall be less than the Excess by which the Cone EFGHN exceeds the Solid X. Let these be the Segments that stand on EO, OF, FP, PG, GR, RH, HS, SE; then the remaining Pyramid whose Base is the Polygon EOFPGRHS, and Vertex the Point N, is greater than the Solid X. Also let the Polygon ATBYCVDQ be described in the Circle ABCD, similar and alike situate to the Polygon EOFPGRHS; upon which erect a Pyramid having the same Altitude as the Cone; and let LBT be one of the Triangles containing the Pyramid, whose Base is the Polygon ATBYCVDQ, and Vertex the Point L; as likewise NFO one of the Triangles containing the Pyramid EOFPGRHS, and Vertex the Point N, and let KT, MO, be joined. Then because the Cone ABCDL is similar to the Cone EFGHN, it shall be as BD is to FH, so is the Axis KL to the Axis MN; but as BD is to FH, so is * BK to FM; and as BK is to FM, conse- * 15. 5.
quently so is KL to MN; and (by Alternation) as BK is to KL, so is FM to MN. And since each is perpendicular, and the Sides about the equal Angles BKL, FMN, are proportional, the Triangle BKL shall be † similar to the Triangle FMN. Again, be- † 6. 6.
cause BK is to KT, as FM is to MO, the Sides are proportional about equal Angles BKT, FMO, for the Angle BKT is the same Part of the four Right Angles at the Center K, as the Angle FMO is of the

four Right Angles at the Center M:) the Triangle
* 6. 6. BKT, shall be * similar to the Triangle FMO; and because it has been proved that BK is to KL, as FM is to MN, and BK is equal to KT, and FM to MO, it shall be as TK is to KL, so is OM to MN; and the proportional Sides are about equal Angles TKL, OMN, for they are Right Angles. Therefore the Triangle LKT shall be similar to the Triangle MNO. And since, by the Similarity of the Triangles BKL, FMN, it is as LB is to BK, so is NF to FM; and, by the Similarity of the Triangles BKT, FMO, it is as KB is to BT, so is MF to FO; it shall be (by Equality of Proportion) as LB is to BT, so is NF to FO. Again, since by the Similarity of the Triangles LTK, NOM, it is as LT is to TK, so is NO to OM; and, by the Similarity of the Triangles KBT, OMF, it is as KT is to TB; so is MO to OF. It shall be (by Equality of Proportion) as LT is to TB, so is NO to OF: But it has been proved that TB is to BL, as OF is to FN. Wherefore, again (by Equality of Proportion) as TL is to LB, so is ON to NF; and therefore the Sides of the Triangles LTB, NOF, are proportional; and so the Triangles LTB, NOF, are equiangular and similar to each other. And consequently the Pyramid, whose Base is the Triangle BKT, and Vertex the Point L, is similar to the Pyramid whose Base is the Triangle FMO, and Vertex the Point N; for they are contained under similar Planes equal in Multitude: But
† 8 of this. similar Pyramids that have triangular Bases, are † to one another in the triplicate Proportion of their homologous Sides. Therefore the Pyramid BKTL to the Pyramid FMON has a triplicate Proportion of that which BK has to FM. In like Manner, drawing Right Lines from the Points A, Q, D, V, C, Y to K, as also others, from the Points E, S, H, R, G, P, to M, and erecting Pyramids on the Triangles having the same Vertices as the Cones, we demonstrate that every Pyramid of one Cone, to every one of the other Cone, has a triplicate Proportion of that which the Side BK has to the homologous Side MF, that is, which BD has to FH. But as one of the Antecedents is to one of the Consequents, so are ‡ all the
‡ 12. 5. Antecedents to all the Consequents. Therefore as the

the Pyramid BKTL is to the Pyramid EMON, ſo is the whole Pyramid whoſe Baſe is the Polygon AT BYCVDQ, and Vertex the Point L, to the whole Pyramid, whoſe Baſe is the Polygon EOFPGRHS, and Vertex the Point N. Wherefore the Pyramid, whoſe Baſe is the Polygon ATBYCVDQ, and Vertex the Point L, to the Pyramid whoſe Baſe is the Polygon EOFPGRHS, and Vertex the Point N, has a triplicate Proportion of that which BD hath to FH. But the Cone whoſe Baſe is the Circle ABCD, and Vertex the Point L, is ſuppoſed to have to the Solid X a triplicate Proportion of that which BD has to FH. Therefore as the Cone, whoſe Baſe is the Circle ABCD, and Vertex the Point L, is to the Solid X, ſo is the Pyramid whoſe Baſe is the Polygon ATBYCVDQ, and Vertex the Point L, to the Pyramid whoſe Baſe is the Polygon EOFPRHS, and Vertex the Point N. But the ſaid Cone is greater than the Pyramid that is in it, for it comprehends it. Therefore the Solid X alſo is greater than the Pyramid, whoſe Baſe is the Polygon EOFPGRHS, and Vertex the Point N; but it is alſo leſs, which is abſurd. Therefore the Cone, whoſe Baſe is the Circle ABCD, and Vertex the Point L, to ſome Solid leſs than the Cone, whoſe Baſe is the Circle EFGH, and Vertex the Point N, has not a triplicate Proportion of that which BD has to FH. In like Manner, we demonſtrate that the Cone EFGHN, to ſome Solid leſs than the Cone ABCDL, has not a triplicate Proportion of that which FH has to BD. Laſtly, I ſay the Cone ABCDL, to a Solid greater than the Cone EFGHN, has not a triplicate Proportion of that which BD has to FH: For, if this be poſſible, let it be ſo to ſome Solid Z greater than the Cone EFGHN. Then (by Inverſion) the Solid Z, to the Cone ABCDL, has a triplicate Proportion of that which FH has to BD. But ſince the Solid Z is greater than the Cone EFGHN, the Solid Z ſhall be to the Cone ABCDL, as the Cone EFGHN, is to ſome Solid leſs than the Cone ABCDL; and therefore the Cone EFGHN, to ſome Solid leſs than the Cone ABCDL, hath a triplicate Proportion of that which FH has to BD, which has been proved to be impoſſible. Therefore the Cone ABCDL, to

ſome Solid greater than the Cone EFGHN, has not a triplicate Proportion of that which BD has to FH. It has been alſo demonſtrated, that the Cone ABCDL, to ſome Solid leſs than the Cone EFGHN, hath not a triplicate Proportion of that which BD has to FH. Wherefore the Cone ABCDL, to the Cone EFGHN, has a triplicate Proportion of that which BD has to FH. But as Cone is to Cone, ſo is † Cylinder to Cylinder. For a Cylinder having the ſame Baſe as a Cone, and the ſame Altitude is ‡ triple of the Cone, ſince it is demonſtrated, that every Cone is one third Part of a Cylinder, having the ſame Baſe and equal Altitude. Wherefore alſo a Cylinder to Cylinder has a triplicate Proportion of that which BD has to FH. Therefore, *ſimilar Cones and Cylinders are to one another in a triplicate Proportion of the Diameters of their Baſes*; which was to be demonſtrated.

† 15. 5.
‡ 10 *of this*.

PROPOSITION XIII.

THEOREM.

If a Cylinder be divided by a Plane parallel to the oppoſite Planes; then as one Cylinder is to the other Cylinder, ſo is the Axis to the Axis.

LET the Cylinder AD be divided by the Plane GH, parallel to the oppoſite Planes AB, CD, and meeting the Axis EF in the Point K. I ſay, as the Cylinder BG is to the Cylinder GD, ſo is the Axis EK to the Axis KF.

For let the Axis EF be both ways produced to L and M, and put any Number of EN, NL, &c. each equal to the Axis EK; and any Number of FX, XM, &c. each equal to FK. And thro' the Points L, N, X, M, let Planes parallel to ABCD paſs. And in thoſe Planes from L, N, X, M, as Centers, deſcribe the Circles OP, RS, TY, VQ, each equal to AB, CD, and conceive the Cylinders PR, RB, DT, TQ, to be compleated. Then becauſe the Axis LN, NE, EK, are equal to each other, the Cylinders PR, RB, BG will be * to one another as their Baſes. And therefore the Cylinders PR, RB, BG, are equal. And ſince the Axis LN, NE, EK, are

* 11 *of this*.

are equal to each other, as also the Cylinders PR, RB, BG; and the Number of LN, NE, EK, is equal to the Number of PR, RB, BG: The Axis KL shall be the same Multiple of the Axis EK, as the Cylinder PG, is of the Cylinder GB. For the same Reason, the Axis MK is the same Mutiple of the Axis KF, as the Cylinder GQ is of the Cylinder GD. Now, if the Axis KL be equal to the Axis KM, the Cylinder PG shall be equal to the Cylinder GQ; if the Axis LK be greater than the Axis KM, the Cylinder PG shall be likewise greater than the Cylinder GQ; and if less, less. Therefore, because there are four Magnitudes, *viz.* the Axis EK, KF, and the Cylinders BG, GD, and there are taken their Equimultiples, namely, the Axis KL and the Cylinder PG, the Equimultiples of the Axis EK, and the Cylinder BG; and the Axis KM, and the Cylinder GQ, the Equimultiples of the Axis KF, and the Cylinder GD: And it is demonstrated, that if the Axis LK exceeds the Axis KM, the Cylinder PG will exceed the Cylinder GQ; and if it be equal, equal, and less, less. Therefore, as the Axis EK is to the Axis KF, so † is the Cylinder BG to the Cylinder, GD. Wherefore, *if a Cylinder be divided by a Plane parallel to the opposite Planes, then as one Cylinder is to the other Cylinder, so is the Axis to the Axis*; which was to be demonstrated. † Def. 5. 5.

PROPOSITION XIV.

THEOREM.

Cones and Cylinders being upon equal Bases, are to one another as their Altitudes.

LET the Cylinder EB, FD, stand upon equal Bases AB, CD. I say, as the Cylinder EB is to the Cylinder FD, so is the Axis GH to the Axis KL.

For produce the Axis KL to the Point N; and put LN, equal to the Axis GH; and let a Cylinder CM be conceived about the Axis LN. Then because the Cylinders EB, CM, have the same Altitude, they are * to one another as their Bases. But * 11 of this.

their

their Bases are equal. Therefore the Cylinders EB, CM, will be also equal. And becaufe the Cylinder FM is cut by a Plane CD, parallel to the oppofite Planes, it fhall be as the Cylinder CM is to the Cylinder FD, fo is the Axis LN, to the Axis KL. But the Cylinder CM is equal to the Cylinder EB; and the Axis LN to the Axis GH. Therefore the Cylinder EB is to the Cylinder FD, as the Axis GH is to the Axis KL. And as the Cylinder EB is to the Cylinder FD, fo is ‡ the Cone ABG to the Cone CDK; for the Cylinders are * triple of the Cones. Therefore, as the Axis GH is to the Axis KL, fo is the Cone ABG to the Cone CDK, and fo the Cylinder EB to the Cylinder FD. Wherefore, *Cones and Cylinders being upon equal Bafes, are to one another as their Altitudes*; which was to be demonftrated.

‡ 15. 5.
* 10 of this.

PROPOSITION XV.

THEOREM.

The Bafes and Altitudes of equal Cones and Cylinders are reciprocally proportional; and Cones and Cylinders, whofe Bafes and Altitudes are reciprocally proportional, are equal to one another.

LET the Bafes of the equal Cones and Cylinders, be the Circles ABCD, EFGH, and their Diameters AC, EG; and Axis KL, MN; which are alfo the Altitudes of the Cones and Cylinders: And let the Cylinders AX, EO, be compleated. I fay, the Bafes and Altitudes of the Cylinders AX, EO, are reciprocally proportional, that is, the Bafe ABCD is to the Bafe EFGH, as the Altitude MN is to the Altitude KL.

For, the Altitude KL is either equal to the Altitude MN, or not equal. Firft, let it be equal; and the Cylinder AX, is equal to the Cylinder EG. But Cylinders and Cones that have the fame Altitude, are * to one another as their Bafes. Therefore the Bafe ABCD is equal to the Bafe EFGH. And confequently, as the Bafe ABCD is to the Bafe EFGH, fo is the Altitude MN to the Altitude KL.

* 11 of this.

But

But if the Altitude KL be not equal to the Altitude MN, let MN be the greater. And take PM equal to LK from MN; and let the Cylinder EO be cut thro' P by the Plane TYS, parallel to the opposite Planes of the Circles EFGH, RO, and conceive ES to be a Cylinder, whose Base is the Circle EFGH, and Altitude PM. Then, because the Cylinder AX is equal to the Cylinder EO, and ES is some other Cylinder, the Cylinder AX to the Cylinder ES, shall be as the Cylinder EO, is to the Cylinder ES. But as the Cylinder AX is to the Cylinder ES, so is * the Base ABCD to the Base EFGH; for the Cylinders AX, ES have the same Altitude. And as the Cylinder EO is to the Cylinder ES, so is † the Altitude MN to the Altitude MP; for the Cylinder EO, is cut by the Plane TYS parallel to the opposite Planes. Therefore, as the Base ABCD is to the Base EFGN, so is the Altitude MN to the Altitude MP. But the Altitude MP is equal to the Altitude KL. Wherefore as the Base ABCD is to the Base EFGH, so is the Altitude MN to the Altitude KL. And therefore, the Bases and Altitudes of the equal Cylinders AX, EO, are reciprocally proportional.

* 11 *of this.*

† 13 *of this.*

And if the Bases and Altitudes of the Cylinders AX, EO, are reciprocally proportional, that is, if the Base ABCD be to the Base EFGH, as the Altitudes MN is to the Altitude KL. I say, the Cylinder AX is equal to the Cylinder EO. For the same Construction remaining; because the Base ABCD is to the Base EFGH, as the Altitude MN is to the Altitude KL; and the Altitude KL is equal to the Altitude MP. It shall be as the Base ABCD is to the Base EFGH, so is the Altitude MN to the Altitude MP. But as the Base ABCD is to the Base EFGH, so is the Cylinder AX to the Cylinder ES; for they have the same Altitude. And as the Altitude MN is to the Altitude MP, so is ‡ the Cylinder EO to the Cylinder ES. Therefore, as the Cylinder AX is to the Cylinder ES, so is the Cylinder EO to the Cylinder ES. Wherefore the Cylinder AX is equal to the Cylinder EO. In like manner we prove this in Cones; *which was to be demonstrated.*

‡ 11 *of this.*

PRO-

PROPOSITION XVI.

PROBLEM.

Two Circles being about the same Center, to inscribe in the greater a Polygon of equal Sides even in Number, that shall not touch the lesser Circle.

LET ABCD, EFGH, be two given Circles about the Center K. It is required to inscribe a Polygon of equal Sides even in Number in the Circle ABCD, not touching the lesser Circle EFGH.

Draw the Right Line BD through the Center K, as also AG, from the Point G at Right Angles to
* 16. 3. BD, which produce to C; this Line will * touch the Circle EFGH. Then bisecting the Circumference BAD, and again bisecting the half thereof, and doing this continually, we shall have a Circumference left at last less then AD. Let this Circumference be LD, and draw LM from the Point L perpendicular to BD, which produce to N; and join LD,
† 29. 3. DN. And then LD is † equal to DN. And since LN is parallel to AC, and AC touches the Circle EFGH, LN will not touch the Circle EFGH. And much less do the Right Lines LD, DN, not touch the Circle. And if Right Lines, each equal to LD, be applied round the Circle ABCD, we shall have a Polygon inscribed therein of equal Sides, even in Number that does not touch the lesser Circle EFG, *which was to be demonstrated.*

PROPOSITION XVII.

PROBLEM.

To describe a solid Polyhedron, in the greater of two Spheres, having the same Center, which shall not touch the Superficies of the lesser Sphere.

LET two Spheres be supposed about the same Center A. It is required to describe a solid Polyhedron

lyhedron in the greater Sphere, not touching the Superficies of the lesser Sphere.

Let the Spheres be cut by some Plane passing thro' the Center. Then the Sections will be Circles; for because a Sphere is * made by the turning of a Semicircle about the Diameter which is at rest: In whatsoever Position the Semicircle is conceived to be, the Plane in which it is shall make a Circle in the Superficies of the Sphere. It is also manifest that this Circle is a great Circle, since the Diameter of the Sphere, which is likewise the Diameter of the Semicircle, is † greater than all Right Lines that are drawn in the Circle, or Sphere. Now, let BCDE be that Circle of the greater Sphere, and FGH of the lesser Sphere; and let BD, CE be two of their Diameters drawn at Right Angles to one another. Let BD meet the lesser Circle in the Point G, from which to AG let GL be drawn at Right Angles, and AL joined. Then bisecting the Circumference EB, as also the half thereof, and doing thus continually, we shall have left at last a certain Circumference less than that Part of the Circumference of the Circle BCD, which is subtended by a Right Line equal to GL. Let this be the Circumference BK. Then the Right Line BK is less than GL; and BK shall be the Side of a Polygon of equal Sides, even in Number, not touching the lesser Circle. Now, let the Sides of the Polygon in the Quadrant of the Circle BE, be the Right Lines BK, KL, LM, ME; and produce the Line joining the Points K, A, to N: And raise ‡ AX from the Point A perpendicular to the Plane of the Circle BCDE, meeting the Superficies of the Sphere in the Point X, and let Planes be drawn thro' AX, and BD, and thro' AX, and KN, which from what has been said will make great Circles in the Superficies of the Sphere. And let BXD, KXN, be Semicircles on the Diameters BD, KN. Then because XA is perpendicular to the Plane of the Circle BCDE, all Planes that pass thro' XA shall also * be perpendicular to that same Plane. Therefore the Semicircles BXD, KXN are perpendicular to that same Plane. And because the Semicircles BED, BXD, KXN, are equal; for they stand upon equal Diameters BD, KN; their Quadrants BE, BX, KX, shall be also

equal

* Def. 14. 11.

† 15. 3.

‡ 12. 11.

* 18. 11.

equal. And therefore as many Sides as the Polygon in the Quadrant BE has, so many Sides may there be in the Quadrants BX, KX, equal to the Sides BK, KL, LM, ME. Let those Sides be BO, OP, PR, RX, KS, ST, TY, YX: And join SO, TP, YR; and let Perpendiculars be drawn from O, S, to the Plane of the Circle BCDE.
† 38. 11. These will fall † on BD, KN, the common Sections of the Planes; Because the Planes of the Semicircles BXD, KXN, are perpendicular to the Plane of the Circle BCDE. Let the said Perpendiculars be OV, SQ, and join VQ. Then, since the equal Circumferences BO, SK, are taken in the equal Semicircles BXD, KXN, and OV, SQ are Perpendiculars, OV shall be equal to SQ, and BV to KQ. But the whole BA is to the whole KA. Therefore the Part remaning VA, is equal to the Part remaining QA. Therefore as BV is to VA, so is KQ to
‡ 2. 6. QA: And so VQ is ‡ parallel to BK. And since OV and SQ are both perpendicular to the Plane of
* 6. 11. the Circle BCDE, OV shall be * parallel to SQ. But it has also been proved equal to it. Wherefore
† 33. 1. QV, SO, are † equal and parallel. And because QV is parallel to SO, and also parallel to KB, SO shall be
‡ 9. 11. also ‡ parallel to KB: But BO, KS, join them.
* 7. 11. Therefore KBOS is * a quadrilateral Figure in one Plane: For if two Right Lines be parallel, and Points be taken in both of them, a Right Line joining the said Points is in the same Plain as the Parallels are. And for the same Reason, each of the quadrilateral Figures SOPT, TPRY, are in one Plane.
† 2. 11. And the Triangle YRX, is † in one Plane. Therefore, if Right Lines be supposed to be drawn from the Points O, S, P, T, R, Y, to the Point A, there will be constituted a certain solid polyhedrous Figure within the Circumferences BX, KX, composed of Pyramids, whose Bases are the quadrilateral Figures RBOS, SOPT, TPRY, and the Triangle YRX; and Vertices the Point A. And if there be made the same Constructionon each of the Sides KL, LM, ME, like as we have done on the Side KB, as also in the other three Quadrants, and the other Hemisphere, there will be constituted a polyhedrous Figure described in the Sphere, composed

of

of Pyramids whose Bases are the aforesaid quadrilateral Figures, and the Triangle YRX, being of the same Order, and Vertices the Point A. I say, the said Polyhedron does not touch the superficies of the Sphere, wherein the Circle FGH is. Let AZ be drawn ‡ from the Point A, perpendicular to the Plane of the quadrilateral Figure KBSO, meeting it in the Point Z, and join BZ, ZK. Then since AZ is perpendicular to the Plane of the quadrilateral Figure KBSO, it shall also be * perpendicular to all Right Lines that touch it, and are in the same Plane. Wherefore AZ is perperdicular to BZ and ZK. And because AB is equal to AK, the Square of AB shall be also equal to the Square of AK: And the Squares of AZ, ZB are † equal to the Square of AB. For the Angle at Z is a Right Angle. And the Squares of AZ, ZK, are equal to the Square of AK. Therefore the Squares of AZ, ZB, are equal to the Squares of AZ, ZK. Let the common Square of AZ be taken away. And then the Square of BZ remaining, is equal to the Square of ZK remaining: And so the Right Line BZ is equal to the Right Line ZK. After the same Manner we demonstrate that Right Lines drawn from the Point Z to the Points O, S, are each equal to BZ, ZK. Therefore a Circle described about the Center Z, with either of the Distances ZB, ZK, will also pass thro' the Points O, S. And because BKSO is a quadrilateral Figure in a Circle, and OB, BK, KS, are equal, and OS is less than BK; the Angle BZK shall be obtuse; and so BK greater than BZ. But GL also is much greater than BK. Therefore GL is greater than BZ. And the Square of GL is greater than the Square of BZ. And since AL is equal to AB, the Square of AL shall be equal to the Square of AB. But the Squares of AG, GL, together, are equal to the Square of AL, and the Squares of BZ, ZA, together, equal to the Square of AB: Therefore the Squares of AG, GL, together, are equal to the Squares of BZ, ZA, together: But the Square of BZ is less than the Square of GL: Therefore the Square of ZA is greater than the Square of AG; And so the Right Line ZA will be greater than the Right Line AG. But AZ is perpendicular to one Base of the Polyhedron, and

‡ 11. 11.

* Def. 3. 11.

† 47. 1.

and A G to the Superficies. Wherefore the Polyhedron does not touch the Superficies of the lesser Sphere. Therefore, *there is described a solid Polyhedron in the greater, of two Spheres having the same Center, which doth not touch the Superficies of the lesser Sphere*; which was to be demonstrated.

Coroll. Also if a solid Polyhedron be described in some other Sphere, similar to that which is described in the Sphere BCDE; the solid Polyhedron described in the Sphere BCDE, to the solid Polyhedron described in that other Sphere, shall have a triplicate Proportion of that which the Diameter of the Sphere BCDE hath to the Diameter of that other Sphere. For the Solids being divided into Pyramids, equal in Number and of the same Order, the said Pyramids shall be similar. But similar Pyramids are to each other in a triplicate Proportion of their homologous Sides. Therefore the Pyramid whose Base is the quadrilateral Figure KBOS, and Vertex the Point A, to the Pyramid of the same Order into the other Sphere, has a triplicate Proportion of that which the homologous Side of one, has to the homologous Side of the other, that is, which AB, drawn from the Center A of the Sphere, to that Line which is drawn from the Center of the other Sphere. In like Manner, every one of the Pyramids, that are in the Sphere whose Center is A, to every one of the Pyramids of the same Order in the other Sphere, hath a triplicate Proportion of that which AB has to that Line drawn from the Center of the other Sphere. And as one of the Antecedents is to one of the Consequents, so are all the Antecedents to all the Consequents. Wherefore the whole solid Polyhedron, which is in the Sphere described about the Center A, to the whole solid Polyhedron that is in the other Sphere, hath a triplicate Proportion of of that which AB hath to the Line drawn from the Center of the other Sphere, that is, which the Diameter BD has to the Diameter of the other Sphere.

PRO-

PROPOSITION XVIII.

THEOREM.

Spheres are to one another in a triplicate Proportion of their Diameters.

SUPPOSE ABC, DEF, are two Spheres, whose Diameters are BC, EF. I say, the Sphere ABC to the Sphere DEF has a triplicate Proportion of that which BC has to EF.

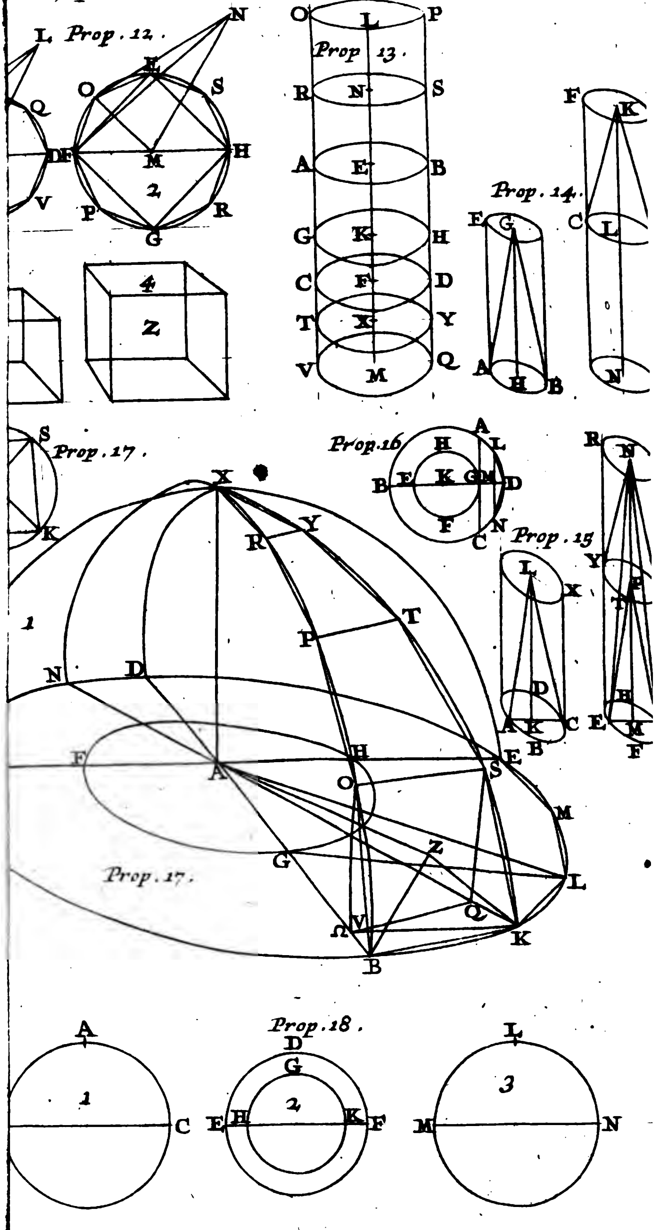
For if it be not so, the Sphere ABC to a Sphere either lesser or greater than DEF, will have a triplicate Proportion of that which BC has to EF. First, let it be to a lesser as GHK. And suppose the Sphere DEF to be described about the Sphere GHK; and let there be described * a solid Polyhedron in the greater Sphere DEF, not touching the Superficies of the lesser Sphere GHK; also let a solid Polyhedron be described in the Sphere ABC, similar to that which is described in the Sphere DEF. Then the solid Polyhedron in the Sphere ABC, to the solid Polyhedron in the Sphere DEF, will have † a triplicate Proportion of that which BC has to EF: But the Sphere ABC to the Sphere GHK, hath a triplicate Proportion of that which BC hath to EF. Therefore as the Sphere ABC is to the Sphere GHK, so is the solid Polyhedron in the Sphere ABC to the solid Polyhedron in the Sphere DEF; and (by Inversion) as the Sphere ABC is to the solid Polyhedron that is in it, so is the Sphere GHK to the solid Polyhedron that is in the Sphere DEF; but the Sphere ABC is greater than the solid Polyhedron that is in it. Therefore the Sphere GHK is also greater than the solid Polyhedron that is in the Sphere DEF, and also less than it, as being comprehended thereby, which is absurd. Therefore the Sphere ABC to a Sphere less than the Sphere DEF, hath not a triplicate Proportion of that which BC has to EF. After the same Manner it is demonstrated that the Sphere DEF to a Sphere less than ABC, has not a triplicate Proportion of that which EF has to BC. I say, moreover, that the Sphere ABC to a Sphere greater than DEF, hath not a triplicate

* 17 of this.

† Cor. to the last Prop.

plicate Proportion of that which BC has to EF; for if it be possible, let it have to the Sphere LMN greater than DEF. Then (by Inversion) the Sphere LMN to the Sphere ABC, shall have a triplicate Proportion of that which the Diameter EF has to the Diameter BC; but as the Sphere LMN is to the Sphere ABC, so is the Sphere DEF to some Sphere less than ABC, because the Sphere LMN is greater than DEF. Therefore the Sphere DEF to a Sphere less than ABC, hath a triplicate Proportion to that which EF has to BC, which is absurd, as has been before proved. Therefore the Sphere ABC to a Sphere greater than DEF, has not a triplicate Proportion of that which BC has to EF. But it has also been demonstrated, that the Sphere ABC to a Sphere less than DEF, has not a triplicate Proportion of that which BC has to EF. Therefore the Sphere ABC to the Sphere DEF, has a triplicate Proportion of that which BC has to EF; *which was to be demonstrated.*

FINIS.


Prop. 12.
Prop 13.
Prop. 14.
Prop. 17.
Prop. 16
Prop. 15
Prop. 17.
Prop. 18.

THE ELEMENTS

Of *Plain* and *Spherical*

TRIGONOMETRY;

Alſo a Short

TREATISE

OF THE

NATURE and ARITHMETICK

OF

LOGARITHMS.

THE ELEMENTS Of *Plain* and *Spherical* TRIGONOMETRY.

DEFINITIONS.

THE Business of Trigonometry is to find the Angles when the Sides are given, and the Sides, or the Ratio's of the Sides, when the Angles are given, and to find Sides and Angles, when Sides and Angles are given: In order to which, it is necessary that not only the Peripheries of Circles, but also certain Right Lines in and about Circles be supposed divided into some determined Number of Parts.

And so the Ancient Mathematicians thought fit to divide the Periphery of a Circle into 360 *Parts (which they call Degrees;) and every Degree into* 60 *Minutes, and every Minute into* 60 *Seconds: And again, every Second into* 60 *Thirds, and so on. And every Angle is said to be of such a Number of Degrees and Minutes, as there are in the Arc measuring that Angle.*

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There are some that would have a Degree divided into centesimal Parts, rather than sexagesimal ones: And it would perhaps be more useful to divide, not only a Degree, but even the whole Circle in a decuple Ratio; which Division may some time or other gain Place. Now, if a Circle contains 360 *Degrees, a Quadrant thereof, which is the Measure of a Right Angle, will be* 90 *of those Parts: And if it contains* 100 *Parts, a Quadrant will be* 25 *of these Parts.*

The Complement *of an Arc is the Difference thereof from a Quadrant.*

A Chord, *or* Subtense, *is a Right Line drawn from one End of the Arc to the other.*

The Right Sine *of any Arc, which also is commonly called only a Sine, is a Perpendicular falling from one End of an Arc, to the Radius drawn thro' the other End of the said Arc. And is therefore the Semisubtense of double the Arc,* viz. $DE = \frac{1}{2} DO$, *and the Arc* DO *is double of the Arc* DB. *Hence, the Sine of an Arc of* 30 *Degrees, is equal to the one half of the Radius. For (by* 15. *El.* 4.) *the Side of an Hexagon inscribed in a Circle, that is, the Subtense of* 60 *Degrees is equal to the Radius. A Sine divides the Radius into two Segments* CE, EB; *one of which,* CE, *which is intercepted between the Center and the Right Sine, is the Sine Complement of the Arc* DB *to a Quadrant, (for* $CE = FD$ *which is the Sine of the Arc* DH,) *and is called the* Cosine. *The other Segment* BE, *which is intercepted between the Right Sine and the Periphery, is called a* versed Sine, *and sometimes a* Sagitta.

And if the Right Line CG *be produced from the Center* C, *thro' one End* D *of the Arc, until it meets the Right Line* BG, *which is perpendicular to the Diameter drawn thro' the other End* B *of the Arc, then* CG *is called the* Secant, *and* BG *the* Tangent *of the Arc* DB.

The Cosecant and Cotagent of an Arc is the Secant or Tangent of that Arc, which is the Complement of the former Arc to a Quadrant. Note, *As the Chord of an Arc, and of its Complement to a Circle, is the same; so likewise is the Sine, Tangent, and Secant of an Arc the same as the Sine, Tangent, and Secant of m-plement to a Semicircle.*

The

The Sinus Totus *is the greatest Sine, or the Sine of* 90 *Degrees, which is equal to the Radius of the Circle.*

A Trigonometrical Canon *is a Table, which, beginning from one Minute, orderly expresses the Lengths that every Sine, Tangent, and Secant, have in respect of the Radius, which is supposed Unity; and is conceived to be divided* 10,000,000, *or more decimal Parts. And so the Sine, Tangent, or Secant of any Arc, may be had by help of this Table; and contrarywise, a Sine, Tangent, or Secant, being given, we may find the Arc it expresses.* Take Notice, *That in the following Tract,* R *signifies the Radius,* S *a Sine,* Cos. *a Cosine,* T *a Tangent, and* Cot. *a Cotangent.*

The CONSTRUCTION of the *Trigonometrical Canon,*

PROPOSITION I.

THEOREM.

The two Sides of any Right-angled Triangle being given, the other Side is also given.

For (by 47 of the first *Element*) $ACq = ABq + BCq$ and $ACq - BCq = ABq$. and interchangeably $ACq - ABq = BCq$. Whence, by the Extraction of the square Root, there is given $AC = \sqrt{ABq + BCq}$ and $AB = \sqrt{ACq - BCq}$. And $BC = \sqrt{ACq - ABq}$.

PROPOSITION II.

PROBLEM.

The Sine DE of the Arc DB, and the Radius CD, being given, to find the Cosine DF.

The Radius CD and the Sine DE, being given in the Right-angled Triangle CDE, there will be given (by the last *Prop.*) $\sqrt{CDq - DEq} = DF$.

PROPOSITION III.

PROBLEM.

The Sine DE of any Arc DB being given, to find DM or BM the Sine of half the Arc.

DE being given, CE (by the last *Prop.*) will be given, and accordingly EB which is the Difference between the Cosine and Radius. Therefore DE, EB, being given in the Right-angled Triangle DBE, there will be given DB, whose half DM is the Sine of the Arc DL = $\frac{1}{2}$ the Arc BD.

PROPOSITION IV.

PROBLEM.

The Sine BM of the Arc BL being given, to find the Sine of double that Arc.

THE Sine BM being given, there will be given (by *Prop.* 2.) the Cosine CM. But the Triangles CBM, DBE, are equiangular, because the Angles at E and M are Right Angles, and the Angle at B common. Wherefore (by 4. 6.) we have CB : CM : : BD, or 2 BM : DE. Whence, since the three first Terms of this Analogy are given, the fourth also, which is the Sine of the Arc DB, will be known.

Coroll. Hence, CB : 2 CM : : BD : 2 DE, that is, the Radius is to double the Cosine of one half of the

the Arc DB, as the Subtense of the Arc DB, is to the Subtense of double that Arc. Also CB : 2CM : : (2BM : 2DE : :) BM : DE : : ½ CB : CM. Wherefore the Sine of any Arc, and the Sine of its Double being given, the Cosine of the Arc it self is given.

PROPOSITION V.

The Sines of two Arcs BD, FD, being given, to find FI the Sine of the Sum, as likewise EL, the Sine of their Difference.

LET the Radius CD be drawn, and then CO is the Cosine of the Arc FD, which accordingly is given, and draw OP thro' O parallel to DK. Also let OM, GE, be drawn parallel to CB. Then because the Triangles CDK, COP, CHI, FOH, FOM, are equiangular. In the first Place, CD : DK : : CO : OP, which Consequently is known. Also we have CD : CK : : FO : FM, and so likewise this shall be known. But because FO = EO, then will FM = MG = ON. And so OP + FM = FI = Sine of the Sum of the Arcs : And OP — FM, that is, OP — ON = EL = Sine of the Difference of the Arcs. W.W.D.

Coroll. Because the Differences of the Arcs BE, BD, BF, are equal, the Arc BD shall be an Arithmetical Mean between the Arcs BE, BF.

PROPOSITION VI.

The same Things being supposed, Radius is to double the Cosine of the mean Arc as the Sine of the Difference, to the Difference of the Sines of the Extreams.

FOR we have CD : CK : : FO : FM, whence by doubling the Consequents CD : 2CK : : FO : 2FM, or to FG; which is the Difference of the Sines EL, FI. W.W.D.

Coroll.

Coroll. If the Arc BD be 60 Degrees, the Difference of the Sines FI, EL, shall be equal to the Sine, FO; of the Distance. For in this Case, CK, is the Sine of 30 Degrees, Double thereof being equal to Radius; and so since CD = 2 CK, we shall have FO = FG. And conseqently, if the two Arcs BE, BF, are Equidistant from the Arc of 60 Degrees, the Difference of the Sines shall be equal to the Sine of the Distance FD.

Coroll. 2. Hence, if the Sines of all Arcs be given distant from one another by a given Interval, from the Beginning of a Quardrant to 60 Degrees, the other Sines may be found by one Addition only. For the Sine of 61 Degrees = Sine of 59 Degrees + Sine of 1 Degree. And the Sine of 62 Degrees = Sine of 58 Degrees + Sine of 2 Degree. Also the Sine of 63 Degrees = Sine of 57 Degrees + Sine of 3 Degrees, and so on.

Coroll. 3. If the Sines of all Arcs, from the Beginning of a Quadrant to any Part of the Quadrant, distant from each other, by a given Interval be given, thence we may find the Sines of all Arcs to the Double of that Part. For Example, Let all the Sines to 15 Degrees be given; then, by the precedent Analogy, all the Sines to 30 Degrees, may be found. For Radius is to double the Cosine of 15 Degrees, as the Sine of 1 Degree, is to the Difference of the Sines of 14 Degrees, and 16 Degrees; so also is the Sine of 3 Degrees, to the Difference between the Sines of 12 and 18 Degrees; and so on continually until you come to the of 30 Degrees.

After the same Manner, as Radius is to double the Cosine of 30 Degrees, or to double the Sine of 60 Degrees, so is the Sine of 1 Degree to the Difference of the Sines of 29 and 31 Degrees : : Sine 2 Degrees, to the Difference of the Sines of 28 and 32 Degrees : : Sine 3 Degrees, to the Difference of the Sines of 27 and 33 Degrees. But in this Case, Radius is to double the Cosine of 30 Degrees, as 1 to $\sqrt{3}$. And accordingly, if the Sines

Sines of the Distances from the Arc of 30 Degrees, be multiply'd by $\sqrt{3}$, the Differences of the Sines will be had.

So likewise may the Sines of the Minutes in the Begining the Quadrant be found, by having the Sines and Cosines of one and two Minutes given. For as the Radius is to double the Cosine of 2' :: Sine 1' : Difference of the Sines of 1' and 3' : : Sine 2' : Difference of the Sines of 0' and 4', that is, to the Sine of 4'. And so the Sines of the four first Minutes being given; we may thereby find the Sines of the others to 8', and from thence to 16', and so on.

PROPOSITION VII.

THEOREM.

In small Arces, the Sines and Tangents of the same Arcs are nearly to one another, in a Ratio of Equality.

FOR because the Triangles CE D, CBG, are equiangular, CE : CB : : ED :: BG. But as the Point E approaches B, EB will vanish in Respect of the Arc BD: Whence CE will become nearly equal to CB. And so ED will be also nearly equal to BG. If EB be less than the $\frac{1}{10,000,000}$ Part of the Radius, then the Difference between the Sine and the Tangent will be also less than the $\frac{1}{10,000,000}$ Part of the Tangent.

Coroll. Since any Arc is less than the Tangent, and greater than its Sine, and the Sine and Tangent of a very small Arc, are nearly equal; it follows that the Arc shall be nearly equal to its Sine; and so in very small Arcs it shall be, as Arc is to Arc, so is Sine to Sine.

PRO-

PROPOSITION VIII.

To find the Sine of the Arc of one Minute.

THE Side of a Hexagon inscribed in a Circle, that is, the Subtense of 60 Degrees, is equal to the Radius, (*by* 15*th* of the 4*th*,) and so the half of the Radius shall be the Sine of the Arc 30 Degrees. Wherefore the Sine of the Arc of 30 Degrees being given, the Sine of the Arc of 15 Degrees may be found, (*by Prop.* 3.) Also the Sine of the Arc of 15 Degrees being given, (*by the same Prop.*) we may have the Sine of 7 Degrees 30 Minutes: So likewise can we find the Sine of the half of this, *viz.* 3 Degrees 45^{I}; and so on, until twelve Bisections being made, we come to an Arc of 52^{2}, 44^{3}, 03^{4}, 45^{5}, whose Cosine is nearly equal to the Radius, in which Case (*as is manifest from Prop.* 7.) Arcs are propotional to their Sines: And so, as the Arc of 52^{2}, 44^{3}, 3^{4}, 45^{5}, is to an Arc of one Minute, so shall the Sine before found, be to the Sine of an Arc of one Minute, which therefore will be given. And when the Sine of one Minute is found, then (*by Prop.* 2. *and* 4.) the Sine and Cosine of two Minutes will be had.

PROPOSITION IX.

THEOREM

If the Angle BAC, *being in the Periphery of a Circle, be bisected by the Right Line* AD, *and if* AC *be produced until* DE = AD *meets it in* E: *then shall* CE = AB.

IN the quadrilateral Figure ABDC (*by* 13. 1.) the Angles B and ACD are equal to two Right Angles = DCE + DCA (*by* 22. 3.) Whence the the Angle B = DCE. But likewise the Angle E = DAC (*by* 5. 1.) = DAB and DC = DB. Wherefore the Triangles BAD and CED are congruous, and CE is equal to AB. W. W. D.

PRO-

PROPOSITION. X.

THEOREM.

Let the Arcs AB, BC, CD, DE, EF, *&c. be equal; and let the Subtenses of the Arcs* AB, AC, AD, AE, &c. *be drawn; then will* AB : AC : : AC : AB + AD : : AD : AC + AE : : AE : AD + AF : : AF : AE + AG.

LET AD be produced to H, AE to I, AF to K, and AG to L, that the Triangles ACH, ADI, AEK, AFL, be Isosceles ones; then because the Angle BAD is bisected, we shall have DH = AB, (*by the last Prop.*) so likewise shall EI = AC, FK = AD, also GL = AB.

But the Isosceles Triangles ABC, CHA, DAI, EAK, FAL, because of the equal Angles at the Bases, are equiangular. Wherefore it shall be as AB : AC : : AC : AH = AB + AD : : AD : AI = AC + AE : : AE : AK = AD + AF : : AF : AL = AE + AG. W.W.D.

Coroll. 1. Because AB is to AC, as Radius is to double the Cosine of $\frac{1}{2}$ the Arc AB, it shall also be (by *Corol. Prop.* 5.) as Radius is to double the Cosine of $\frac{1}{2}$ the Arc AB, so is $\frac{1}{2}$ AB : $\frac{1}{2}$ AC : : $\frac{1}{2}$ AC : : $\frac{1}{2}$ AB + $\frac{1}{2}$ AD : : $\frac{1}{2}$ AD : $\frac{1}{2}$ AC + $\frac{1}{2}$ AE : : $\frac{1}{2}$ AE + $\frac{1}{2}$ AD + $\frac{1}{2}$ AF, *&c.* Now let each of the Arcs AB, BC, CD, *&c.* be 2′; then will $\frac{1}{2}$ AB be the Sine of one Minute, $\frac{1}{2}$ AC the Sine of 2′ Minutes, $\frac{1}{2}$ AD the Sine of 3′ Minutes; $\frac{1}{2}$ AE the Sine of 4′, *&c.* Whence if the Sines of one and two Minutes be given, we may easily find all the other Sines in the following Manner.

Let the Cosine of the Arc of one Minute, that is, the Sine of the Arc of 89 Deg. 59′, be called Q, and make the following Analogies; R : 2 Q : : Sin. 2′ : S. 1′ + S. 3′. Wherefore the Sine of 3 Minutes will be given. Also R : 2 Q : : S. 3′ : S. 2′ + S. 4′. Wherefore the S. 4′ is given; and R. 2 Q : : S. 4′ : S. 3′ + S. 5′; and so the Sine of 5′ will be had.

Like-

Likewise R : 2Q : : S. 5′ : S. 4′+S. 6′; and so we shall have the Sine of 6′. And in like Manner, the Sines of every Minute of the Quadrant will be given. And because the Radius, or the first Term of the Analogy is Unity, the Operations will be with great Ease and Expedition calculated by Multiplication, and contracted by Addition. When the Sines are found to 60 Degrees; all the other Sines may be had by Addition only, (by *Cor.* 1. *Prop.* 5.)

The Sines being given, the Tangents and Secants may be found from the following Analogies, (in the Figure for the Definitions;) because the Triangles CED, CBG, CHI, are equiangular, we have

CE : ED : : CB : BG; that is, Cos. : S : : R : T.
GB : BC : : CH : HI; that is, T : R : : R : Cot.
CE : CD : : CB : CG; that is, Cos. : R : : R : Secant.
DE : CD : : CH : CI; that is, S : R : : R : Cosec.

SCHOLIUM.

That great Geometrician and incomparable Philosopher, Sir Isaac Newton, *was the first that laid down a Series converging*, in infinitum; *from which, having the Arcs given, their Sines may be found. Thus if an Arc be called* A, *and the Radius be an Unit; the Sine thereof will be found to be*

$$A-\frac{A^3}{1.2.3}+\frac{A^5}{1.2.3.4.5}-\frac{A^7}{1.2.3.4.5.6.7}+\frac{A^9}{1.2.3.4.5.6.7.8.9}\ \&c.$$

And the Cosine,

$$1-\frac{A^2}{1.2}+\frac{A^4}{1.2.3.4}-\frac{A^6}{1.2.3.4.5.6}+\frac{A^8}{1.2.3.4.5.6.7.8}\ \&c.$$

These Series in the Beginning of the Quadrant when the Arc A *is but small, soon converge. For in the Series for the Sine, if* A *does not exceed* 10 *Minuites, the two first Terms thereof,* viz. $A-\frac{1}{6}A^3$ *gives the Sine to* 15 *Places of Figures. If the Arc* A *be not greater than one Degree, the three first Terms will exhibit the Sine to* 15 *Places of Figures; and so the said Series are very*

very usefull for finding the first and last Sines of the Quadrant. But the greater the Arc A *is, the more are the Terms of the Series required to have the Sine in Numbers true to a given Place of Figures. And then when the Arc is nearly Equal to the Radius, the Series Convergeres very slow. And therefore, to remedy this I have devised other Series, similar to the* Newtonian *one, wherein, I suppose, the Arc, whose Sine is sought, is the Sum and Difference of two Arcs,* viz. $A+z$, *or* $A-z$: *And let the Sine of the Arc* A, *be called* a, *and the Cosine* b. *Then the Sine of the Arc* $A+z$ *will be expressed thus :*

$$1.\quad a+\frac{bz}{1}-\frac{az^2}{1.2}-\frac{bz^3}{1.2.3}+\frac{az^4}{1.2.3.4}+\frac{bz^5}{1.2.3.4.5}\ \&c.$$

And the Cosine is,

$$2.\quad b-\frac{az}{1}-\frac{bz^2}{1.2}+\frac{az^3}{1.2.3}+\frac{bz^4}{1.2.3.4}-\frac{az^5}{1.2.3.4.5}-\frac{bz^6}{1.2.3.4.5.6}\ \&c.$$

In like Manner the Sine of the Arc $A-z$ *is*

$$3.\quad a-\frac{bz}{1}-\frac{az^2}{1.2}+\frac{bz^3}{1.2.3}+\frac{az^4}{1.2.3.4}-\frac{bz^5}{1.2.3.4.5}-\frac{az^6}{1.2.3.4.5.6}\ \&c.$$

And the Cosine is,

$$4.\quad b+\frac{az}{1}-\frac{bz^2}{1.2}-\frac{az^3}{1.2.3}+\frac{bz^4}{1.2.3.4}+\frac{az^5}{1.2.3.4.5}\ \&c.$$

The Arc A *is an Arithmetical Meen between the Arc* $A-z$ *and* $A+z$. *And the Difference of the Sines are,*

$$5.\quad \frac{bz}{1}-\frac{az^2}{1.2}-\frac{bz^3}{1.2.3}+\frac{az^4}{1.2.3.4}+\frac{bz^5}{1.2.3.4.5}-\frac{az^6}{1.2.3.4.5.6}\ \&c.$$

$$6.\quad \frac{bz}{1}+\frac{az^2}{1.2}-\frac{bz^3}{1.2.3}-\frac{az^4}{1.2.3.4}+\frac{bz^5}{1.2.3.4.5}+\frac{az^6}{1.2.3.4.5.6}\ \&c.$$

Whence the Difference of the Differences or second Difference.

$$7.\ \textit{Produce}\ \frac{2az^2}{1.2}-\frac{2az^4}{1.2.3.4}+\frac{2az^6}{1.2.3.4.5.6}\ \&c.$$

$$\textit{Or,}\ 2a+\frac{z^2}{1.2}-\frac{z}{1.2.3.4}+\frac{z^6}{1.2.3.4.5.6}\ \&c.$$

which

Which Series is equal to double the Sine of the Mean Arc, drawn into the versed Sine of the Arc z, *and Converges very soon. So that if* z *be the first Minute of the Quadrant, the first Term of the Series gives the second Difference to* 15 *Places of Figures, and the second Term to* 25 *Places.*

From hence, if the Sines of the Arcs distant one Minute from each other be given. The Sines of all the Arcs that are in the same Progression may be found by an exceeding easy Operation.

In the first and second Series, if A $=$ o; *then shall* a $=$ c, *and* b *its Cosine, will become Radius, or* 1. *And hence, if the Terms wherein* a *is, are taken away, and* 1 *be put instead of* 6, *the Series, will become the* Newtonian. *In the third and fourth Series, if* A *be* 90 *Degrees, we shall have* b $=$ o, *and* a $=$ 1. *Whence again, taking away all the Terms wherein* b *is, and putting* 1 *instead of* a, *we shall have the* Newtonian *Series arise.*

Note, *All the said Series easily flow from the* Newtonian *ones. By the fifth* Propsition.

PROPOSITION XI.

In a Right-angled Triangle, if the Hypothenuse be made the Radius, then are the Sides the Sines of their opposite Angles; and if either of the Legs be made the Radius, then the other Leg is the Tangent of its opposite Angle, and the Hypothenuse is the Secant of that Angle.

IT is manifest that CB is the Sine of the Arc CD, and AB the Cosine thereof; but the Arc CD is the Measure of the Angle A, and the Complement of the Measure of the Angle C. Moreover, if AB, in the Figure to this Proposition, be supposed Radius, then BC is the Tangent, and AC the Secant of the Arc BD, which is the Measure of the Angle A. So also if BC be made the Radius, then is BA the Tangent, and AC the Secant of the Arc BE, or Angle C. W.W.D.

There-

Therefore as AC being taken as some given Measure, is to BC taken in the same Measure; so shall the Number 10000000 Parts in which the Radius is supposed to be divided, be to a Number expressing in the same Parts the Length of the Sine of the Angle A; that is, it will be

as AC : BC : R : S, A,
by the same Reason, as AC : BA : R : S, C.
also as AB : BC : R : T, A.
and BC : BA : R : T, C.

And so if any three of these Proportionals be given, the fourth may be found by the *Rule of Three.*

PROPOSITION XII.

The Sides of Plain Triangles are as the Sines of their opposite Angles.

IF the Sides of a Triangle, inscribed in a Circle, be bisected by perpendicular Radii, then shall the half Sides be the Sines of the Angles at the Periphery; for the Angle BDC at the Center, is double of the Angle BAC at the Periphery; (by 20 *El. lib.* 3.) and so the half of every of them, *viz,* BDE = BAC, and BE is the Sine thereof. For the same Reason, BF shall be the Sine of the Angle BCA, and AG the Sine of the Angle ABC.

In a Right-angled Triangle we have BD = ½ BC = Radius (*by* 31 *Eucl.* 3.) but Radius is the Sine of a Right Angle: Whence ½ BC is the Sine of the Angle A.

In an Obtuse-angled Triangle, let BL, CL, be drawn, and then the Angle L shall be the Complement of the Angle A to two Right Angles, (*by* 22 *El.* 3.) and so they shall both have the same Sine. But the Angle BDE, (whose Sine is BE) = Angle L. Therefore BE shall be the Sine of the Angle BAC. And so in every Triangle, the Halves of the Sides are the Sines of the opposite Angles; but it is manifest that the Sides are to one another as their Halves. W.W.D.

PROPOSITION XIII.

In a plain Triangle, the Sum of the Legs, the Difference of the Legs, the Tangent of the half Sum of the Angles at the Base, and the Tangent of one half their Difference, are proportional.

LET there be a Triangle ABC, whose Legs are AB, BC, and Base AC. Produce AB to H, so that BH=BC, then shall AH be the Sum of the Legs; and if you make BI=BA, then IH will be the Difference of the Legs. Also the Angle HBC =Angles A+ACB, (*by* 32. *El.* 1.) and so EBC the half thereof = half the Sum of the Angles A and ACB, and its Tangent (putting the Radius=EB) is EC. Again, let BD be drawn parallel to AC, and make HF=CD. Then since HB=CB, we shall have (*by* 4 *El.* 1.) the Angle HBF=CBD=BCA. (*by* 29 *El.* 1.) Also the Angle HBD=Angle A; whence FBD shall be the Difference of the Angles A and ACB; and EBD, whose Tangent is ED, half their Difference. Let IG be drawn thro' I parallel to AC or BD, and then (*by* 2 *El.* 6.) AB: BI :: CD : DG, but AB=BI; whence we shall have CD=DG, but CD=HF, and so HF=DG, and consequently, HG=DF, and $\frac{1}{2}$ HG=$\frac{1}{2}$ DF =DE; and because the Triangles AHC, IHG, are equiangular, it shall be as AH : IH :: HC : HG :: $\frac{1}{2}$ HC : $\frac{1}{2}$ HG :: EC : ED. That is, AH the Sum of the Legs, to IH the Difference of the Legs, shall be as EC the Tangent of one half the Sum of the Angles at the Base, to ED the Tangent of one half their Difference. W. W. D.

PROPOSITION XIV.

In a plain Triangle, the Base, the Sum of the Sides, the Difference of the Sides, and the Difference of the Segments of the Base, are proportional.

LET DC be the Base of the Triangle BCD, about the Center B, with the Radius BC, let a Circle be described. Produce DB to G, and from B let

let fall BE perpendicular to the Base; then shall DG = DB + BC = Sum of the Sides, and DH = Difference of the Sides; and DE, CE, are the Segments of the Base whose Difference is DF; because (*by Cor. Prop.* 38. *El.* 3.) the Rectangle under DC and DF, is equal to the Rectangle under DG, DH, it shall be (*by* 16 *El.* 6.) as DC : DG :: DH : DF.

PROBLEM.

The Sum and Difference of any two Quantities being given, to find the Quantities themselves.

IF one half of the Sum be added to one half of the Difference, the Aggregate shall be equal to the greater of the Quantities; and if from one half of the Sum be taken from one half of the Difference, the Residue shall be equal to the lesser of the Quantities. For let there be two Quantities AB, BC, and let there be taken AD = BC, then DB will be their Difference, and AC their Sum; which, bisected in E, gives AE or EC the half Sum, and DE or EB the half Difference. Hence AB = AE + EB = the half Sum + the half Difference, and BC = CE − EB = the half Sum − the half Difference.

In any plain Triangle if two Angles be given, the third Angle is also given, because it is their Complement to two Right Angles.

If one of the acute Angles of a Right-angled Triangle be given, the other acute Angle will be given, because it is the Complement of the given Angle to a Right Angle.

And if two Sides of a Right-angled Triangle be given, the other Side may be found by the first Proportion without a Canon.

The Trigonometrical Solutions of a Right-angled Triangle, may be as follow. Vid. Fig. A.

	Given	Sought	Make as
1	The Legs A B and BC	The Angles.	AB : BC : : R : T of the Angle A, whose Complement is the Angle C.
2	The Leg AB and the Hypothenuse A C.	The Angles.	AC: AB : : R : S, C whose Complement is the Angle A.
3	The Leg AB and the Angle A.	The other Side BC, and the Hypothenuse AC	R : T, A : : AB : BC. S, C : R : : AB : AC.
4	The Hypothenuse A C, and the Angle C.	The Leg A B.	R : : S, C : : AC : AB.

The Trigonometrical Solutions of oblique angled Triangles. Vid. Fig. to Prop. 12.

	Given	Sought	Make as
1	The Angles A, B, C, and the Side AB.	The Sides BC and AC.	S, C : S A : : AB : BC. Also S, C : S, B : : AB : AC : But when two Angles are given, the third is is also given; whence the Case wherein two Angles and a Side are given, to find the rest, falls into this Case.

	Given	Sought	Make as
2	All the Angles A,B,C.	All the Sides AB, AC, BC.	S, C : S, A : : AB : BC. And S, C : S, B : : AB : BC : Whence if the Angles are given, the Proportions of the Sides may be found, but not the Sides themselves, unless one of them be first known.
3	The two Sides AB, BC, and C the Angle opposite to one of them.	The Angles A and B.	AB : BC : : S, C : S, A; which therefore may be found. When AB the Side opposite to C, the given Angle is longer than BC the Side opposite to the sought Angle, the sought Angle is less than a right one. But when it is shorter, because the Sine of an Angle, and that of its Complement to two Right Angles, is the same, the Species of the Angle A must be first known, or the Solution will be ambiguous.
4	The two Sides AB, BC, and the interjacent Angle B.	The Angles A and C.	*Vid. Fig.* to *Prop.* 13. BC+AB : BC—AB : : T, $\frac{A+C}{2}$: T, $\frac{A-C}{2}$ Whence is known the Difference of the Angles A and C, whose Sum is given; and so (by the *Prob.* following *Prop.* 14.) the Angles themselves will be given.
5	All the Sides AB, BC, AC.	The Angles.	*Vid. Fig.* B. Let the Perpendicular be drawn from the Vertex to the Base, and find the Segments of the Base by *Prop.* 14. *viz.* Make as BC : AC+AB : : AC—AB : DC—DB. And so BD, DC, are given from this Analogy; and thence the Angles ABD, ADC, will be given by the Resolution of Right-angled Triangles.

THE ELEMENTS OF Spherical TRIANGLES.

DEFINITIONS.

I. *THE Poles of a Sphere are two Points in the Superficies of the Sphere that are the Extremes of the Axis.*

II. *The Pole of a Circle in a Sphere, is a Point in the Superficies of the Sphere, from which all Right Lines that are drawn to the Circumference of the Circle, are equal to one another.*

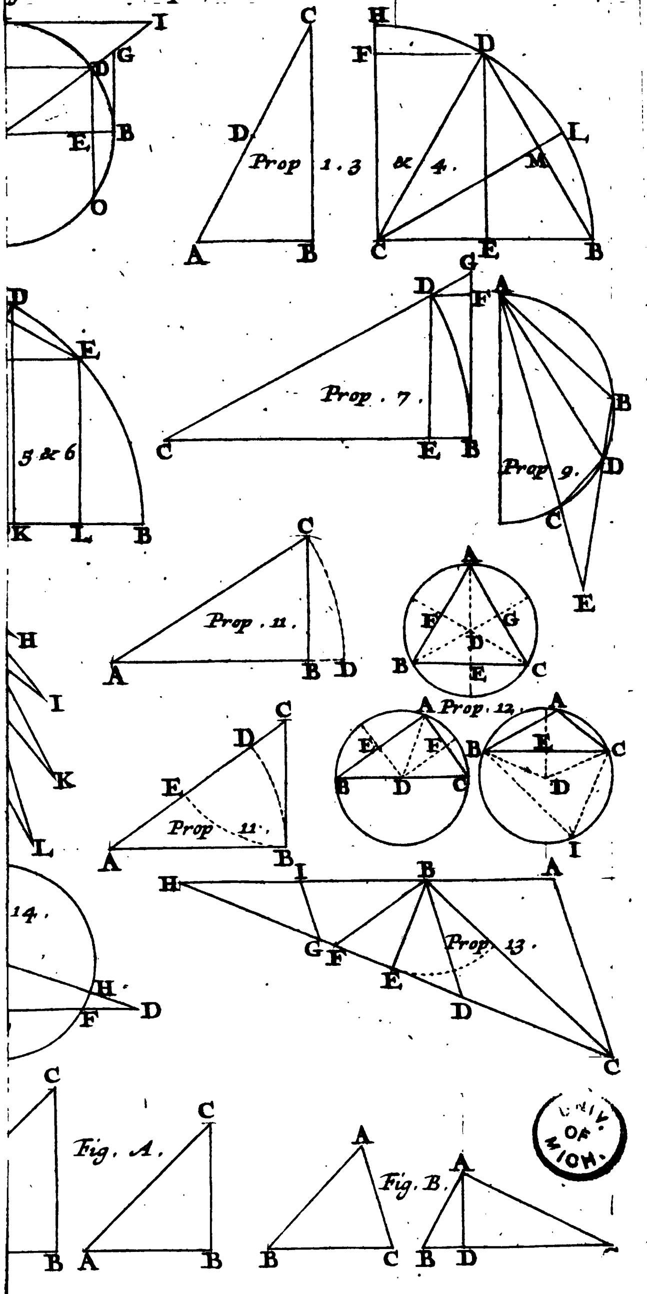
III. *A great Circle in a Sphere, is that whose Plane passes thro' the Center of the Sphere, and whose Center is the same of that of the Sphere.*

IV. *A spherical Triangle is a Figure comprehended under the Arcs of three great Circles in a Sphere.*

V. *A spherical Angle is that which, in the Superficies of the Sphere, is contained under two Arcs of great Circles; and this Angle is equal to the Inclination of the Planes of the said Circles.*

PRO-

efinitions and Prop. 2.



PROPOSITION I.

Great Circles A C B, A F B, mutually bisect each other.

FOR since the Circles have the same Center, their common Section shall be a Diameter of each Circle, and so will cut them into two equal Parts.

Coroll. Hence the Arcs of two great Circles in the Superficies of the Sphere, being less than Semicircles, do not comprehend a Space; for they cannot, unless they meet each other in two opposite Points in a Semicircle.

PROPOSITION II.

If from the Pole C of any Circle AFB, be drawn a Right Line C D to the Center thereof, the said Line will be perpendicular to the Plane of that Circle. Vid. Fig. to Prop. 1.

LET there be drawn any Diameters EF, GH, in the Circle AFB; then because in the Triangles CDF, CDE, the Sides CD, DF, are equal to the Sides CD, DE, and the Base CF equal to the Base CE; (*by Def.* 2.) then (*by* 4 *El.* 1.) shall the Angle CDF=Angle CDE; and so each of them will be a Right Angle. After the same Manner we demonstrate that the Angles CDG, CDH, are Right Angles; and so (*by* 4 *El.* 11.) CD shall be perpendicular to the Plane of the Circle AFE. W.W.D.

Coroll. 1. A great Circle is distant from its Pole by the Interval of a Quadrant; for since the Angles CDG, CDF, are Right Angles, the Measures of them, *viz.* the Arcs CG, CF, will be Quadrants.

2. Great Circles that pass thro' the Pole of some other Circle, make Right Angles with it; and contrari-

wise, if great Circles make Right Angles with some other Circle, they shall pass thro' the Poles of that other Circle, for they must necessarily pass thro' the Right Line DC.

PROPOSITION III.

If a great Circle ECF be described about the Pole A; then the Arc CF intercepted between AC, AF, is the Measure of the Angle CAF, or CDF, Vid. Fig. to Prop. 1.

THE Arcs AC, AF, (by *Cor.* 1. *Prop.* 2.) are Quadrants, and consequently the Angles ADC, ADF, are Right Angles. Wherefore (by *Def.* 6. *El.* 11.) the Angle CDF, (whose Measure is the Arc CF) is equal to the Inclination of the Planes ACB, AFB, and also equal to the spherical Angle CAF, or CBF. W.W.D.

Coroll. 1. If the Arcs AC, AF, are Quadrants, than shall A be the Pole of the Circle passing thro' the Points C and F; for AD is at Rignt Angles to the Plane FDC. (by 14 *El* 11.)

2. The vertical Angles are equal, for each of them is equal to the Inclination of the Circles; also the adjoining Angles are equal to two Right Angles.

PROPOSITION IV.

Triangles shall be equal and congruous, if they have two Sides equal to two Sides, and the Angles comprehending the two Sides also equal.

PROPOSITION V.

Also Triangles shall be equal and congruous, if one Side together with the adjacent Angles in one Triangle, be equal to one Side, and the adjacent Angles of the other Triangle.

PRO-

PROPOSITION. VI.

Triangles mutually Equilateral, are also mutually equiangular.

PROPOSITION. VII.

In Isosceles Triangles, the Angles at the Base are equal.

PROPOSITION VIII.

And if the Angles at the Base be equal, then the Triangle shall be Isosceles.

These four last Propositions are demonstrated in the same Manner, as in plain Triangles.

PROPOSITION IX.

Any two Sides of a Triangle are greater than the third.

FOR the Arc of a great Circle, is the shortest Way, between any two Points in the Superficies of the Sphere.

PROPOSITION X.

A Side of a spherical Triangle is less than a Semicircle.

LET AC, AB, the Sides of the Triangles ABC be produced till they meet in D; then shall the Arc ACD, which is greater than the Arc AC, be a Semicircle.

PROPOSITION XI.

The three Sides of a spherical Triangle are less than a whole Circle.

FOR DB+DC is greater than BC, (by *Prop.* 9.) and adding on each Side BA+AC, DBA+DCA; that is, a whole Circle will be greater than AB+

AB+BC+AC, which are the three Sides of the spherical Triangle ABC.

PROPOSITION XII.

In any spherical Triangle ABC, *the greater Angle* A *is subtended by the greater Side.*

MAKE the Angle BAD = Angle B; then shall AD = BD, (*by* 8 *of this*;) and so BDC = DA + DC, and these Arcs are greater than AC. Wherefore the Side BC, that subtends the Angle BAC, is greater than the Side AC, that subtends the Angle B.

PROPOSITION XIII.

In any spherical Triangle ABC, *if the Sum of the Legs* AB *and* BC *be greater, equal, or less, than a Semicircle, then the internal Angle at the Base* AC *shall be greater, equal, or less, than the external and opposite Angle* BCD, *and so the Sum of the Angles* A *and* ACB *shall also be greater, equal, or less, than two Right Angles.*

FIRST, let AB+BC = Semicircle = AD, then shall BC = BD, and the Angles BCD and D equal, (*by* 8 *of this*,) and therefore the Angle BCD shall be = Angle A.

Secondly, Let AB+BC be greater than ABD; then shall BC be greater than BD; and so the Angle D (that is, the Angle A, by 12 of this) shall be greater than the Angle BCD. In like Manner we demonstrate, if AB+BC be together less than a semicircle, that the Angle A will be less than the Angle BCD. And because the Angles BCD and BCA, are = two Right Angles; if the Angle A be greater than the Angle BCD, then shall A and BCA, be greater than two Right Angles; if the Angle A = BCD, then shall A and BCA be equal to two Right Angles. And if A be less than BCD, then will A and BCA be less than two Right Angles. W. W. D.

PRO-

PROPOSITION XIV.

In any spherical Triangle GHD, *the Poles of the Sides being joined by great Circles, do constitute another Triangle* XMN, *which is the Supplement of the Triangle* GHD, *viz. the Sides* NX, XM, *and* NM, *shall be Supplements of the Arcs that are the Measures of the Angles* D, G, H, *to the Semicircles; and the Arcs that are the Measures of the Angles* M, X, N, *will be the Supplements of the Sides* GH, GD, *and* HD, *to Semicircles.*

FROM the Poles G, H, D, let the great Circles XCAM, TMNO, XKBN, be described; then because G is the Pole of the Circle XCAM, we shall have GM = Quadrant, (by *Cor.* 1 *Prop.* 2.) and since H is the Pole of the Circle TMO, then will HM be also a Quadrant; and so (by *Cor.* 1. *Prop.* 3.) M shall be the Pole of the Circle GH. In like Manner, because D is the Pole of the Circle XBN, and H the Pole of the Circle TMN, the Arcs DN, HN, will be Quadrants; and so (by *Cor.* 1. *Prop.* 3.) N shall be the Pole of the Circle HD. And because GX, DX, are Quadrants, X will be the Pole of the Circle GD. These Things premised.

Because NK = Quadrant, (*Cor.* 1. *Prop.* 2.) then will NK + XB, that is, NX + KB = two Quadrants, or a Semicircle; and so NX is the Supplement of the Arc KB, or of the Measure of the Angle HDG to a Semicircle. In like Manner, because MC = Quadrant, and XA = Quadrant, then will MC + XA; that is, XM + AC = two Quadrants, or Semicircle; and consequently XM is the Supplement of the Arc AC, which is the Measure of the Angle HGD. Likewise, since MO, NT, are Quadrants, we shall have MO + NT = OT + NM = Semicircle. And therefore NM is the Supplement of the Arc OT, or of the Measure of the Angle GHD, to a Semicircle. W. W. D.

Moreover, because DK, HT, are Quadrants, DK + HT, or KT + HD, are equal to two Quadrants, or a Semicircle. Therefore KT, or the Measure of the Angle XNM, is the Supplement of the Side

Side HD to a Semicircle. After the same Manner it is demonstrated, that O C, the Measure of the Angle XMN, is the Supplement of the Side G H, and B A the Measure of the Angle X, is the Supplement of the Side G D. W. W. D.

PROPOSITION XV.

Equiangular spherical Triangles are also equilateral.

FOR their Supplementals (*by* 14 *of this*) are equilateral; and therefore equiangular also; and so themselves are likewise equilateral, (by *Part* 2. *Prop.* 14.)

PROPOSITION XVI.

The three Angles of a spherical Triangle, are greater than two Right Angles, and less than six.

FOR the three Measures of the Angles G, H, D, together with the three Sides of the Triangle XNM, make three Semicircles, (*by* 14 *of this*;) but the three Sides of the Triangle XNM, are less than two Semicircles, (*by* 11 *of this*.) Wherefore the three Measures of the Angles G, H, D, are greater than a Semicircle; and so the Angles G, H, D, are greater than two Right Angles.

The second Part of the Proposition is manifest, for in every spherical Triangle, the external and internal Angles together, only make six Right Angles; wherefore the internal Angles are less than six Right Angles.

PRO-

PROPOSITION XVII.

If from the Point R, *not being the Pole of the Circle* AFBE, *there fall the Arcs* RA, RB, RG, RV, *of great Circles to the Circumference of that Circle; then the greatest of those Arcs is* RA, *which passes thro' the Pole* C *thereof; and the Remainder of it is the least; and those that are more distant from the greatest are less than those which are nearer to it, and they make an obtuse Angle with the former Circle* AFB, *on the Side next to the greatest Arc.* Vid. Fig. to Prop. 1.

BEcause C is the Pole of the Circle AFB, then shall CD and RS, which is parallel thereto, be perpendicular to the Plane AFB. And if SA, SG, SV, be drawn, then shall SA (by 7 *El.* 3.) be greater than SG, and SG greater than SV. Whence in the Right-angled plain Triangles RSA, RSG, RSV, we shall have RSq + SAq, or RAq, greater than RSq + SGq, or RGq; and so RA will be greater than RG, and the Arc RA greater than the Arc RG. In like Manner, RSq + SGq, or RGq shall be greater than RS + SVq, or RVq; and so RG shall be greater than RV, and the Arc RG greater than the Arc RV. 2*dly*, The Angle RGA is greater than the Angle CGA, which is a Right Angle, (by *Cor. Prop.* 3.) and the Angle RVA is greater than the Angle CVA, which also is a Right Angle. Therefore the Angles RGA, RVA, are obtuse Angles.

PROPOSITION XVIII.

In a spherical Triangle right-angled at A, *the Legs containing the Right Angle, are of the same Affection with the opposite Angles; that is, if the Legs be greater or less than Quadrants, then accordingly will the Angles opposite to them be greater or less than Right Angles.* Vid. Fig. 10. Prop. 1.

FOR if AC be a Quadrant, then will C be the Pole of the Circle AFB, and the Angles AGC, or AVC, will be Right Angles. If the Leg AR be greater than a Quadrant, then shall the Angle AGR be

be greater than a Right Angle, (*by 17 of this*;) and if the Leg AX be less than a Quadrant, the Angle AGX shall be less than a Right Angle.

PROPOSITION XIX.

If two Legs of a right-angled spherical Triangle be of the same Affection; (and consequently the Angles,) that is, if they are both less or both greater than a Quadrant, then will the Hypothenuse be less than a Quadrant. Vid. Fig. to Prop. 1.

IN the Triangle ARV, or BRV, let F be the Pole of the Leg AR, then will RF be a Quadrant, which is greater than RV, (*by 17 of this*.)

PROPOSITION XX.

If they be of a different Affection, then shall the Hypothenuse be greater than a Quadrant. Vid. Fig. to Prop. 1.

FOR in the Triangle ARG, the Hypothenuse RG is greater than RF, which is a Quadrant.

PROPOSITION XXI.

If the Hypothenuse be greater than a Quadrant, then the Legs of the Right Angle, and so the Angles opposite to them, are of the same Affection; but if lesser, of a different Affection. Vid. Fig. to Prop. 1.

THIS Proposition being the Converse of the former ones, easily follow from them.

PROPOSITION XXII.

In any spherical Triangle ABC, *if the Angles at the Base* B *and* C, *be of the same Affection, then the Perpendicular falls within the Triangle; and if they be of a different Affection, the Perpendicular falls without the Triangle.*

IN the first Case, if the Perpendicular does not fall within, let it fall without the Triangle, (as in *Fig.* 2.) then in the Triangle ABP, the Side AP is of the same Affection

Affection with the Angle B. And in like Manner, in the Triangle ACP, AP, is of the same Affection with the Angle ACP. Therefore since ABC, and ACP, are of the same Affection, the Angles ABC, ACB, shall be of a different Affection; which is contrary to the Hypothesis.

In the second Case, if the Perpendicular does not fall without, let it fall within, (as in *Fig*. 1.) Then in the Triangle ABP, the Angle B is of the same Affection with the Leg AP. So likewise, in the Triangle ACP, the Angle C is of the same Affection with AP; and therefore the Angles B and C are of the same Affection; which is contrary to the Hypothesis.

PROPOSITION XXIII.

In spherical Triangles BAC, BHE, *right-angled at* A *and* H, *if the same acute Angle* B *be at the Base* BA, *or* BH, *then the Sines of the Hypothenuses shall be proportional to the Sines of the perpendicular Arcs.*

FOR the Right Lines CD, EF, being perpendicular to the same Plane, are parallel. Also FR, DP, perpendicular to the Radius OB, are likewise parallel; wherefore the Planes of the Triangles EFR, CDP, are also parallel, (by 15 *El.* 11.) Wherefore CP, ER, the common Sections of those Planes, with the Plane passing thro' BE, CO, will be parallel, (by 16 *El.* 11.) Therefore the Triangles CDP, EFR, shall be equiangular. Wherefore CP the Sine of the Hypothenuse BC, is to CD the Sine of the perpendicular Arc CA, as ER the Sine of the Hypothenuse BE to EF, the Sine of the perpendicular Arc EH. W.W.D.

PROPOSITION XXIV.

The same Things being supppofed, AQ, HK, *the Sines of the Bases, are proportional to* IA, GH, *the Tangents of the perpendicular Arcs.*

FOR after the same Manner, as in the last Proposition, we demonstrate that the Triangles QAI, KHG, are equiangular; whence QA : AI :: KH : HG.

PRO-

PROPOSITION XXV.

In a spherical Triangle ABC, *right-angled at* A, *as the Cosine of the Angle* B, *at the Base* BA, *is to the Sine of the vertical Angle* ACB, *so is the Cosine of the Perpendicular to the Radius.*

PREPARATION.

LET the Sides AB, BC, CA, be produced, so that BE, BF, CI, CH, be Quadrants; and from the Poles B and C, draw the great Circles EFDG, IHG, then will the Angles at E, F, I, H, be Right Angles. And so D is the Pole of BAE, (by *Cor.* 2. *Prop.* 2 of this,) and G the Pole of IFCB; also AE will be = Complement of the Arc BA, and FE the Measure of the Angle B = GD, and DF their Complement; Also BC shall be = FI = Measure of the Angle G, and CF their Complement. Likewise CA = HD, and DC their Complement. These Things premised in the Triangles HIC, DCF, right-angled at I and F, and having the same acute Angle C, since BA is less than a Quadrant, it will be as S, DF : S, HI : : S, DC : S, HC; that is, the Cosine of the Angle B, is to the Sine of the vertical Angle BCA, as the Cosine of CA is to Radius. W.W.D.

PROPOSITION XXVI.

The Cosine of the Base : Cosine of the Hypothenuse : : R : Cos. of the Perpendicular.

FOR in the Triangles AED, CFD, right-angled at E, F, having the same acute Angle D; because AE is less than a Quadrant, we have S, EA : S, CF : : S, DA : S, DC. W.W.D.

PRO-

PROPOSITION XXVII.

S, *of the Base* : R : : T, *of the Perpendicular* : T, *of the Angle at the Base.*

FOR in the Triangles BAC, BEF, right-angled at A and E, and having the ſame acute Angle B; becauſe AC is leſs than a Quadrant, we have S, BA : S, BE : : T, AC : T, EF. W.W.D.

PROPOSITION XXVIII.

Coſ. of the vertical Angle : R : : T, *of the Perpendicular* : T, *of the Hypothenuſe.*

IN the Triangles GIF, GHD, right-angled at I and H, and having the ſame acute Angle G, becauſe HD is leſs than HC, or a Quadrant, it is as S, GH : S, GI : : T, HD : T, IF.

PROPOSITION XXIX.

S, *of the Hypothenuſe* : R : : S, *of the Perpendicular* : S, *of the Angle at the Baſe.*

IN the aforeſaid Triangles, we have S, IF : S, GF : : S, HD : S, GD.

PROPOSITION XXX.

R : *Coſ. of the Hypothenuſe* : : T, *of the vertical Angle* : *Cot. of the Angle at the Baſe.*

IN the Triangles HIC, DFC, right-angled at I and F, and having the ſame acute Angle C, becauſe DF is leſs than a Quadrant, we have S, CI : S, CF : : T, HI : T, DF, that is, R : Coſ. BC : : T, C : Cot. B.

The laſt ſix Propoſitions are ſufficient for ſolving all the ſixteen Caſes of Right-angled ſpherical Triangles. Theſe ſixteen Caſes, with their Analogies deduc'd from the ſaid Propoſitions, are as follows:

X Given

	Given besides the Right Angle.	Sought	*Vid. Fig.* to *Prop.* 25, 26, 27, 28, 29, 30.	
1	A C and C.	B.	R : Cos. CA : : S, C : Cos. B of the same Kind with CA.	By the Inverse of Prop. 25.
2	A C and B.	C.	Cos. C A : R : : Cos. B : S, C this is ambiguous.	by Prop. 25.
3	B and C.	A C.	S, C : Cos. B : : R : Cos. CA of the same kind with the Angle B.	by Prop. 25 and 18.
4	B A, C A.	B C.	R : Cos. B A : : Cos. A C : Cos. BC. If B A, A C, be of the same Affection, and not Quadrants, then BC will be less than a Quadrant. If they be of a different Affection, B C shall be greater than a Quadrant.	by Prop. 26. 19, and 20.
5	B A, B C.	A C.	Cos. B A : R : : Cos. B C : Cos. C A. If B C be less than a Quadrant, then shall BA and CA be of the same Affection, if greater, of a different; but B A is given, and therefore the Species thereof. Wherefore the Species of A C is also given.	by Prop. 26 and 21.
6	B A, CA.	B.	S, BA : R : : T, C A : T, B of the same Affection with the opposite Side CA.	by Prop. 27 and 18.
7	BA, B.	AC.	R : S, BA : : T, B : T, AC of the same kind with B.	by Prop. 27 and 18.
8	AC, B.	B A.	T, B : R : : T, CA : S, BA ambiguous.	by Prop. 27.

Given

	Given besides the Right Angle.	Sought		
9	BC, C.	A C	R : Cof. C : : T, B C : T, C A. If B C be less than a Quadrant, the Angles C and B are of the same Affection, if greater, of a different. Therefore if the Species of the Angle B be given, then will A C be given.	By Prop. 28 and 21
10	AC, C.	BC.	Cof. C : R : : T, C A : T, B C. And so if the Angle C, and C A, be of the same Affection, then BC shall be lesser than a Quadrant, if of a different, greater.	by Prop. 28, 20, 21.
11	B C. A C.	C.	T, B C : R : : T, C A : Cof. C. If B C be less than a Quadrant, then C A and B A, and consequently the Angles, shall be of the same Affection, if greater, of a different, but the Species of C A is given. Therefore the Species of the Angle C will be also given.	by Prop. 28, and 21.
12	BC, B.	A C.	R : S, B C : : S, B : S, A C of the same Species with B.	by Prop. 29, and 18.
13	A C, B.	BC.	S, B : S, A C : : R : S, B C ambiguous.	by Prop. 29.
14	B C. A C.	B.	S, BC : R : : S, A C : S, B of the same Species with CA.	by Prop. 29.
15	B, C.	B C.	T, C : R : : Cot. B : Cof. B C. And so if the Angles B and C are of the same Affection, then shall B C be less than a Quadrant, if of a different, greater.	by Prop. 30, 19, and 20.

	Given besides the Right Angle.	fought		
16	BC, C.	B.	R : Cos. BC : : T, C : Cot. B. And so if BC be lesser than a Quadrant, the Angles C and B shall be of the same Affection ; if greater of a different. But the Species of the Angle C is given; therefore the Species of the Angle B will be given also.	by Prop. 30 and 31.

Of

Of the Solution of Right-angled Sperical Triangles, by the five circular Parts.

THE Lord *Napier*, (the Noble Inventor of Logarithms,) by a due Consideration of the Analogies, by which Right-angled spherical Triangles are are solv'd, found out two Rules, easy to be remembered, by means of which, all the sixteen Cases may be solved; for since, in the these Triangles, besides the Right Angle, there are three Sides and two Angles; the two Sides Comprehending the Right Angle, and the Complements of the Hypothenuse, and the two other Angles, were called by *Nepier*, *Circular Parts*. And then there are given any two of the said Parts, and a third is sought; one of these three which is called the *Middle Part*, either lyes between the other two Parts, which are called *Adjacent Extremes*, or is separated from them, and then are called *Opposite Extremes*; so if the Complement of the Angle B (*Fig.* to *Prop*, 25.) be supposed the middle Part, then the Leg A B, and the Complement of the Hypothenuse BC, are adjacent Extremes Parts; but the Complement of the Angle C, and the Side AC, are opposite Extremes, Also if the Complement of the Hypothenuse BC, be supposed the middle Part, then the Complements of the Angles B and C are adjacent Extremes, and the Legs AB, AC, are opposite Extremes. In like Manner, supposing the Leg A B the middle Part, the Complement of the Angle B and A C, are adjacent Extremes; for the Right Angle A does not interrupt the Adjacence, because it is not a circular Part. But the Complement of the Angle C, and the Complement of the Hypothenuse BC, are opposite Extremes to the said middle Part These Things premised.

RULE. I.

In any Right-angled spherical Triangle, the Rectangle under the Radius, and the Sine of the middle Part, is equal to the Rectangle under the Tangents of the adjacent Parts.

RULE II.

The Rectangle under the Radius, and the Sine of the middle Part, is equal to the Rectangle under the Cosines of the opposite Parts,

Each of the Rules have three Cases. For the middle Part may be the Complement of the Angle B, or C, or the Complement of the Hypothenuse BC; or one of the Legs, AB, AC.

Case 1. Let the Complement of the Angle C be the middle Part. Then shall AC, and the Complement of the Hypothenuse BC, be adjacent Extremes By *Prop*. 28. the Cosine of the Vertical Angle C is to Radius, as the Tangent of CA, is to the Tangent of the Hypothenuse BC. Then (by Alternation) we shall have Cos. C : T, CA : : R: T, BC. But R : T, BC : : Cot. BC : R. (as has been before shewn.) Wherefore Cos. C : T, AC : : Cot. BC : R; whence R × Cos. C = T, AC × Cot. BC.

And the Complement of the Angle B, and AB, are opposite Extremes, to the same middle Part, the Complement of the Angle C. (and by *Prop*. 25.) as the Cosine of the Angle C, to the Sine of the Angle CDF, so is the Cosine of DF to Radius. But the Sine of CDF = AE = Cos. BA, and Cos. DF = S, EF = S, Angle B. Whence it will be as Cos. C: Cos. BA : : S, B : R. And R × Cos. C = Cos. BA × S, B; that is, Radius drawn into the Sine of the middle Part, is equal to the Rectangle under the Cosines of the opposite Extremes.

Case 2. Let the Complement of the Hypothenuse BC, be the middle Part; then the Complements of the Angles B and C, will be adjacent Extremes. In the Triangle DCF (by *Prop*. 27.) it is as S, CF : R : : T, DF. T, C. Whence (by Alternation) S, CF : T, DF : : (R : T, C : :) Cot. C : R; But S, CF = Cos. BC and T, DF = Cot. B. Wherefore R × Cos. BC = Cot. C × Cot. B; that is, Radus drawn into the Sine of the middle Part, is equal to the Product of the Tangents, of the adjacent extreme Parts.

And

And AB, AC, are the opposite Extremes to the said middle Part, *viz.* the Complement of BC; and (by Prop. 26.) Cos. BA : Cos. BC :: R : Cos. AC. Wherefore we shall have R × Cos. BC = Cos. BA × Cos. AC.

Case 3. Lastly, Let AB be the middle Part; and then the Complement of the Angle B, and AC will be adjacent Extremes, (and by Prop. 27.) S, AB : R :: T, CA :: TB. Whence, S, AB : T, CA : (R : T, B ::) Cot. B : R. And so R × S, AB = T, CA × Cot. B.

Moreover, the Complements of BC, and the Angle C, are opposite Extremes to the same middle Part AB; and in the Triangle GHD (by Prop. 25.) we have Cos. D : S, DGH :: Cos. GH : R. But Cos. D = Cos. AE = S, AB, and S, G = S, IF = S, BC. Also Cos. GH = S, HI = S, C. Wherefore it will be as S, AB : S, BC :: S, C : R. And hence R × S, AB = S, BC × S, C.

And so in every Case the Rectangle under the Radius, and the Sine of the middle Part, shall be equal to the Rectangle under the Cosines of the opposite Extremes, and to the Rectangle under the Tangents of the adjacent Extremes. And consequently if the aforesaid Equations be resolved into Analogies, (by 16. *El.* 6.) the unknown Parts may be found by the Rule of Proportion. And if the Part sought be the middle one; then shall the first Term of the Analogy be Radius, and the second and third, the Tangents or Cosines of the extreme Parts. If one of the Extremes be sought, the Analogy must begin with the other; and the Radius, and the Sine of the middle Part, must be put in the middle Places, that so the Part sought may be in the fourth Place.

IN Oblique-angled Spherical Triangles (*Fig. to Prop.* 31.) BCD, if a perpendicular Arc AC be let fall from the Angle C to the Base, continued, if need be, so as to make two Right-angled Spherical Triangles BAC, DAC, then by those Right-angled Triangles, may most of the Cases of Oblique-angled ones be solved.

PRO-

PROPOSITION XXXI.

The Cosines of the Angles B *and* D, *at the Base* BD, *are proportional to the Sines of the Vertical Angles* BCA, DCA.

FOr Cos. Angle B : S, BCA : : (Cos. CA : R : :) Cos. D : S, DCA. (*by* 25. *of this.*)

PROPOSITION XXXII.

The Cosines of the Sides BC, DC, *are proportional to the Cosines of the Bases* BA, DA.

FOR Cos. BC : Cos. BA : : (Cos. CA : R : :) Cos. DC : Cos. DA. (*by* 26. *of this.*)

PROPOSITION XXXIII.

The Sines of the Bases BA, DA, *are in a reciprocal Proportion of the Tangents of the Angles* B *and* D *at the Base* BD.

BECAUSE (*by* 27. *of this.*) S, BA : R : : T, AC : T. of the Angle B. And by the same inversely, R : S, DA : : T, of the Angle D : T, AC. Then will it be (by the Equality of purturbate Ratio, according to *Prop.* 23. *El.* 5.) S, BA : S, DA : : T, Angle D : T, Angle B.

PROPOSITION XXXIV.

The Tangents of the Sides BC, DC, *are in a reciprocal Proportion of the Cosines of the Vertical Angles* BCA, DCA,

BECAUSE by alternating the the 28th *Prop.* we have

T, BC : R, : : T, CA : Cos. BCA

and by the same R : Cos. DCA : : T, DC : T, CA.

Wherefore by Equality of perturbate Proportion.

T, BC : Cos. DCA : : T, DC : Cos. BCA.

PRO-

PROPOSITION. XXXV.

The Sines of the Sides BC, DC, *are proportional to the Sines of the opposite Angles* D *and* B.

BECAUSE by the 29th *of this*, S, BC : R :: S, CA : S, of the Angle B. And by the same, inverting R : S, DC :: S, Angle D : S, of CA, whence by Equality of perturbate Ratio, S, BC : S, DC :: S, D : S, B.

PROPOSITION. XXXVI.

In any Spherical Triangle ABC, *the Rectangle* CF × AE, *or* FM × AE, *contained under the Sines of the Legs* BC, BA, *is to the Square of the Radius, as* IL *or* IA — LA *the Difference of the versed Sines of the Base* CA, *and the Difference of the Legs* AM, *to* GN, *the versed Sine of the Angle* B.

LET a great Circle PN be described from the Pole B; and let BP, BN be Quadrants, and then PN is the Measure of the Angle B; also describe from the same Pole B a lesser Circle CFM thro' C; the Planes of these Circles shall be perpendicular to the Plane BON, (*by the* 2*d of this*.) And PG, CH being perpendicular in the same Plane, fall on the common Sections ON, FM; suppose in G, H. Again draw HI, perpendicular to AO, and then the Plane drawn thro' CH, HI, shall be perpendicular to the Plane AOB. Whence AI which is perpendicular to HI will be perpendicular to the Right Line CI, (by *Def.* 4. *El.* 11.) and so AI is the versed Sine of the Arc AC, and AL the the versed Sine of the Arc AM = BM — BA = BC — BA. The Isosceles Triangles CFM, PON, are equiangular, since MF, NO, as also CF, PO (by 16. *El.* 11.) are parallel. Wherefore, if Perpendiculars CH, PG be drawn to the Sides FM, ON, the Triangles will be divided similarly, and we shall have FM : ON :: MH : GN. Also, because the Triangles AOE, DIH, DLM, are equiangular, we shall have AE : AO : IL : MH.

But

But it has been proved that FM : ON :: MH : GN. Wherefore it shall be as AE×FM : AQ×ON :: IL×MH : MH×GN, or as IL to GN, that is, the Rectangle, under the Sines of the Legs, is to the Square of Radius, as the Difference of the versed Sines of the Base, and the Difference of the Legs BC, BA, is to the versed Sine of the Angle B. W. W. D.

PROPOSITION XXXVII.

The Difference of the versed Sines of two Arcs drawn into Radius, is equal to the Rectangle under the Sine of half the Sum and the Sine of half the Difference of those Arcs.

LET there be two Arcs BE, BF, whose Difference EF, let be bisected in D; then shall BD be the half Sum, and FD the half Difference of those Arcs. GE=IL is the Difference of the versed Sines of the Arcs BE, BF; also FO is the Sine of the half Differencs of the Arcs. And because the Triangles CDK, FEG, are equiangular, we have DK : GE :: (CD : FE ::) $\frac{1}{2}$CD : $\frac{1}{2}$FE. Whence DK×$\frac{1}{2}$FE, or DK×FO=GE×$\frac{1}{2}$CD=IL×$\frac{1}{2}$CD. W. W. D.

PROPOSI ION XXXVIII.

The versed Sine of any Arc, drawn into the half the Radius, is equal to the Square of the Sine of one half of the said Arc.

THE Triangles CBM, DEB are equiangular, since the Angles at M and E are Right Angles, and the Angle at B is common. Wherefore EB : BD :: BM : BC. And then will EB×BC=BM×BD; and EB×$\frac{1}{2}$BC=BM×$\frac{1}{2}$BD=BMq. W. W. D.

PRO-

PROPOSITION XXXIX.

In any Spherical Triangle ABC, *whose Legs containing the Angle* B *are* BC, AB, *and Base subtending that Angle* AC: *If the Arc* AM *be taken* = *Difference of the Legs* = BC — AB. *Then shall the Rectangle under the Sines of the Legs* BC, BA, *be to the Square of the Radius, as the Rectangle under the Sine of the Arc* $\frac{AC+AM}{2}$, *and the Sine of the Arc* $\frac{AC-AM}{2}$ *is to the Square of the Sine of one half of the Angle* B. Vid. Fig. to Prop. 36.

BECAUSE the Rectangle under the Sines of the Legs AB, BC, is to the Square of Radius, as IL is to the versed Sine of the Angle B, or as $\frac{1}{2}R \times IL$ to $\frac{1}{2}R$ drawn into the versed Sine of the Angle B (by *Prop.* 36. of this.) And since $\frac{1}{2}R \times IL$ = Rectangle under the Sines of the Arcs $\frac{AC+AM}{2}$, and $\frac{AC-AM}{2}$ (by *Prop.* 37. of this.) And also $\frac{1}{2}R$ drawn into the versed Sine of the Angle B is equal to the Square of the Sine one half of the Angle B. Therefore the Rectangle under the Sines of the Sides to the Square of Radius, shall be as the Rectangle under the Sines of the Arcs $\frac{AC+AM}{2}$ and $\frac{AC-AM}{2}$, is to the Square of the Sine of one half the Angle B. W. W. D.

The

The twelve Cases of Oblique-angled Spherical Triangles are as follow.

In the Original, the Proportion was thus, Cof. BC : R : : T B : Cot, BCA.

This Proposition in the Original was as in the foregoing. The Species of the Angle BCA may be known by Prop. 18 *and* 19.

	Given.	fought.	make as
1	Angles B, D, and BC.	Angle C.	R : Cof. BC : : T, B : Cot. BCA, (by *Prop.* 30. of this.) Alfo Cof. B : S, BCA : : Cof. D : S, DCA (by 31. of this.) Wherefore the Sum of the Angles BCA, DCA, if the Perpendicular falls within the Triangle, or the Difference, if it falls without, will be = BCD. Whether the Perpendicular falls within, or without the Triangle, may be known from the Affection of the Angles B and D (by 22. of this,) which Admonition ought to be obferved in the following Solutions.
2	Angles B, BCD and the Side BC.	Angle D.	R : Cof. BC : : T, B : Cot. BCA (*Prop.* 30. of this.) And S, BCA : S, DCA : : Cof. B : Cof. D (by *Prop.* 31.) If BCA be lefs than BCD, the Angle D fhall be of the fame Affection with the Angle B. If BCA be greater than the Angle BCD, then the Angles B and D, fhall be of a different Affection, by the Converfe of *Prop.* 22.
3	The Sides BC, CD and the Angle B.	The Side BD.	R : Cof. B : : T, BC : T, BA (by 28. of this.) and Cof. BC : : Cof. BA : : Cof. DC : Cof. DA (by 32. of this.) the Sum or Difference of BA or DA, according as the Perpendicular falls within, or without the Triangle, is equal to BL, which cannot be known, unlefs the Species of the Angle D be firft known.

Given.

	Given.	fought.	make as
4	The Sides BC, DB and the Angle B.	The Side CD.	R : Cof. B : : T, BC : T, BA, (by 28. of this.) and Cof. BA : Cof. BC : : Cof. DA : Cof. DC. (by *Prop*. 32. of this.) According as DA is fimilar or difimilar to CA, or to the Angle BDC, fo fhall DC be leffer or greater than a Quadrant, (by 19. and 20 of this.)
5	Angle B, D, and the Side BC.	The Side BD.	R : Cof. B : : T, BC : T, BA, (by 28. of this.) And T, D : T, B : : S, BA : S, DA (by 33. of this.) The Sum or Difference of which BA and DA is = BD.
6	The Sides BC, CD and the Angle B.	Angle D.	R : Cof. B : : T, BC : T, BA, (by *Prop*. 28. of this.) and S, DA : S, BA : : T, B : T, D. (by 33. of this.) According as BD is greater or leffer than BA, the Angle D fhall be fimilar or diffimilar to the Angle B. (by 22. of this.)
7	The Sides BC, DC and the Angle B.	The Side DC.	Cof. BC : R : : Cot. B : T, BCA, (by 30. of this.) and T, DC : T, BC : : Cof. BCA : Cof. DCA, (by 34. of this.) The Sum or Difference of the Angles BCA, DCA, according as the Perpendicular falls within or without the Triangle, is equal to the Angle BCD.
8	The Angle BCD, and the Side BC.	Angle BCD.	Cof. BC : R : : Cot. B : T, BCA, (by 30. of this.) Alfo Cof. DCA : Cof. BCA : : T, BC, T, DC, (by 34. of this.) If the Angle DCA be fimilar to the Angle B (that is, if AD be fimilar to CA,) then DC fhall be lefs than a Quadrant. If the Angles DCA and B be diffimilar, then DC fhall be greater than a Quadrant, which follows (from *Prop*. 18, 19 and 20. of this.)

Given.

	Given.	sought.	make as
9	The Sides BC, DB and the Angle B.	The Angle D.	S, CD : S, B :: S, BC : S, D, which is Ambiguous. The Analogy follows from *Prop.* 35. of this.
10	The Angles B, D, and the Side BC.	The Side DC.	S, D : S, BC :: S, B : S, DC, which Side is Ambiguous.
11	All the Sides AB, BC CA. *Vid. Fig. Prop.* 36.	The Angle B.	As the Rectangle under the Sines of the Legs AB, BC : The Square of Radius :: the Rectangle under the Sines of the Arcs $\frac{AC+AM}{2}$ and $\frac{AC-AM}{2}$: the Square of the Sine of $\frac{1}{2}$ the Angle B. (by *Prop.* 39.
12	All the Angles G, H, D. *Vid. Fig. Prop.* 14.	The Side GD.	In the Triangle XNM, the Arc MN is the Complement of the Angle GHD to a Semicircle. XM is the Complement of the Angle G, and XN the Complement of the Angle D. And the Angle X, the Complement of the Side GD to a Semicircle. Wherefore if the Angles be changed into Sides, and the Sides into Angles, the Operation will be the same, as in *Case* 11. of this, since Arcs and their Complements to Semicircles have the same Sines.

The

The following REMARK by SAMUEL CUN.

THAT this is true but in a particular Case, *viz.* when two of the Angles of the Triangle are Right ones. and two of the Sides Quadrants, may be thus demonstrated. For if possible, let some Triangle RST, *Fig.* to *Prop.* 14th be such, that its Sides RS, ST, TR, be equal to the Measures of GHD, HGD, GDH, the Angles of a Triangle GHD; and also, that the Measures of RST, PTR, TRS, the Angles of the Triangle RST be equal to GH, GD, HD, the Sides of the Triangle GHD. And produce MX, MN, two Sides of the supplemental Triangle to Semicircles, and they will meet somewhere, suppose at E; and there will be constructed thereby the Triangle NEX, of which XE (the Supplement of XM, which, by the 14*th Prop.* was the Supplement of the Measure of the Angle HGD) is equal to the Measure it self of the same Angle HGD: And in like manner, NE (the Supplement of NM, which, by the 14*th Prop.* was the Supplement of the Measure the Angle GHD) is equal to the Measure it self of the same Angle GHD. But the third Side XN, is not the Measure of the third Angle GDH, but its Supplement, by the 14*th Prop.* Moreover, of the Angle EXN (whose Supplement is NXM) the Measure, by the 14*th Prop.* is equal to GD; and of the Angle XNE, (whose Supplement is MNX) the Measure, by the 14th *Prop.* is equal to HD. But of the third NEX, (which is equal to NMX) the Measure is not equal to GH, but its Supplement.

Now make NV = RT = BK, the Measure of the Angle GDH, and draw the great Circle EV. And since RS, by Supposition, is equal to the Measure of the Angle GHD, which is equal to EN; and since the Measure of the Angle SRT, is by Supposition, equal to DH, which is also equal to the Measure of the Angle XNE; the Angle XNE, is equal to the Angle R. Then consequently, by the

4th

4th Prop. the Triangles SRT, ENV, will have the Base ST, equal to the Base EV; the Angle T, to the Angle NVE, and the Angle S, to the Angle NEV. But ST, (which is equal to EV,) by Supposition, is equal to the Measure of the Angle HGD; to which Measure XE is also equal. Therefore, EV is equal to XE; and consequently, by the *7th Prop.* the Angle EVX is equal to the Angle EXV; and the Angle EXV (whose Measure, as hath been shewn above, is equal to GD) is equal to the Angle T, (or NVE,) since by Supposition, the Measure of this is also equal to GD. Therefore the Angle EVX is equal to to the Angle EVN, and so both right ones; and consequently EXV a right one also. Therefore, by the 2d *Cor.* to the 2d *Prop.* EV and EX are both Quadrants.

But if EV be a Quadrant, and at Right Angles to NX, then E, by 2d *Prop.* and its *Coroll.* is the Pole of NX; and so EN a Quadrant also, and the Angle ENV a right one. Therefore, if the Sides of a Triangle (NEV, or its Equal) RST, are equal to the Measures of the Angles of some other Triangle GHD, and the Measures of the Angles of the former, equal to the Sides of the latter; two Sides of such a Triangle RST, or GHD, must be Quadrants, and two Angles of each right ones.

Therefore, if a Triangle RST be constructed whose Sides are equal to the Measures of the Angles another Triangle GHD: The Measures of the Angles of the Triangle RST, shall not be equal to the Sides of the Triangle GHD, unless in the one Case before-mentioned. Therefore the Measures of the Angles of the Triangle GHD, used as the Sides of a Triangle in the 11th Case, will not give us a Side of GHD, but the Measure of an Angle of the Triangle RST, unless in the one afore-mentioned Case; *which was to be demonstrated.*

But to find a Side GD of Spherical Triangle GHD, whose Angles are all given, Produce MN, that Side of the supplemental Triangle, which is equal to the Supplement of the Measure of GHD, the Angle opposite to the Side sought, and MX, either of the other Sides till they meet as in E. And there, as hath been before shewn, the Sides EX, EN, of the Triangle

angle EXN, are exactly equal to the Measures of the Angles HGD, GHD, of the Triangle GHD; and of the Angles EXN, ENX, of the Triangle EXN, the Measures are equal to GD, HD. But the Side XN is equal to the Supplement of the Measure of the Angle GDH. And of the Angle XEN, the Measure is equal to the Supplement of GH.

Therefore the SOLUTION is thus:

Change one of the Angles GDH, adjacent to the Side sought into its Supplement; and then work with the Measures of the Angles as tho' they were Sides, and the Result will be GD, the Side sought.

The preceeding Fault, as well as the Omissions hereafter mention'd, are not peculiar to our Author; but may be found in Dr. *Harris*, Mr. *Caswell*, Mr. *Heynes*, and many other Trigonometrical Writers.

In the Solution of our 8th and 9th Cases, they have told us, that the *Quæsita* are Ambiguous; which sometimes, indeed, is true, but sometimes also false: Therefore, as I conceive it, they ought to have laid down Rules, by help of which we might discover when the *Quæsita* are Ambiguous, and when not.

This Oversight may be corrected by the following Directions: Wherein, because every Sine corresponds to two Arches, to one less than a Quadrant, and to another, which is the Supplement of the former to a Semicircle, (a true Distinction of which, of these are to be used, being necessary to be known, before a proper Solution can be given, to such Problems as these are,) I shall beg leave, for Brevity Sake, to call the lesser Arch the Acute Value, and the greater the Obtuse; whether the Sine be of an Angle or a Side.

In the tenth Case, there are given two Angles B, D, *and* BC *a Side opposite to one of those Angles* D, *to find* DC *the Side opposite to the other.*

TO the Acute Value of DC, and also to its Obtuse one, add BC; and if each of these Sums are

$\left\{\begin{matrix}\text{greater}\\ \text{leſs}\end{matrix}\right\}$ than a Semicircle, when the Sum of the Angles B, D, is $\left\{\begin{matrix}\text{greater}\\ \text{leſs}\end{matrix}\right\}$ than two Right Angles; both the Values of DC may be admitted, and then is Ambiguous: But when only one of thoſe Sums is $\left\{\begin{matrix}\text{greater}\\ \text{leſs}\end{matrix}\right\}$ than a Semicircle, only one Value of DC can be true, *viz.* the $\left\{\begin{matrix}\text{Obtuſe}\\ \text{Acute}\end{matrix}\right\}$ one; and then is not Ambiguous.

In the ninth Caſe there are given two Sides BC, DC, *and one Angle* B, *oppoſite to* DC *one of thoſe Sides, to find* D *the Angle oppoſite to the other.*

TO the Acute Value of D, and alſo to its Obtuſe Value, add B; and if each of theſe Sums is $\left\{\begin{matrix}\text{greater}\\ \text{leſs}\end{matrix}\right\}$ than two Right Angles, when the Sum of the Sides is $\left\{\begin{matrix}\text{greater}\\ \text{leſs}\end{matrix}\right\}$ than a Semicircle, both the Values of D may be admitted, and conſequently, D is Ambiguous: But when only one of thoſe Sums is $\left\{\begin{matrix}\text{greater}\\ \text{leſs}\end{matrix}\right\}$ than two Right Angles, only one Value of D is true, *viz.* the $\left\{\begin{matrix}\text{Obtuſe}\\ \text{Acute}\end{matrix}\right\}$ one; and then not Ambiguous.

Nor are we better uſed in the firſt Propoſition; for tho' it is determined by the given Angles, whether the Perpendicular falls within or without the Triangles, yet in each of thoſe Varieties, the *Quæſita* will be ſometimes Ambiguous, and ſometimes not.

In

In this first Proposition there are given two Angles B, D, *and* BC, *a Side opposite to* D, *one of them, to find* C *the third Angle.*

1. Let the Perpendicular fall within; that is, let the given Angles be of the same Species.

TO the Acute Value of DCA, and also to its Obtuse one, add the Angle BCA; and if each of these Sums is less than two Right Angles, then either the Acute Value of DCA, or its Obtuse one added to BCA, gives a Value of BCD; which, therefore, is Ambiguous. And when only one of these Sums is less than two Right Angles, the Acute Value of DCA, added to BCA, gives the only Value of BCD; which then is not Ambiguous; tho' in both Varieties the Perpendicular fell within.

2. Let the Perpendicular fall without; that is, let the given Angles be of different Species.

WHEN the Obtuse Value of the Angle DCA is less than the Angle BCA, the Angle BCD may be had by substracting either Value of DCA from BCA; and then BCD is Ambiguous. But when the Obtuse Value of DCA is not less than BCA, the Acute Value of DCA, taken from BCA, gives the single Value of BCD; which therefore is not Ambiguous; tho' in both Varieties the Perpendicular fell without.

In the fifth Case, we lie under the same Misfortune, where there are given, as in the first, the Angles B, D, *and the Side* BC, *to find* BD *the Side lying between those given Angles.*

1. When the Perpendicular falls, within; that is, when the given Angles are of the same Species.

TO the Acute Value of DA, and so also to its Obtuse one, add BA; and if each of these Sums is less than a Semicircle, then either the Acute Value of DA, or its Obtuse one, added to BA, gives the

Value of BD; which thence is Ambiguous. And when only one of these Sums is less than a Semicircle, the Acute Value of DA, added to BA, gives the only Value of BD; which then is not Ambiguous; tho' in both Varieties the Perpendicular fell within.

2. When the Perpendicular falls without; that is, when the given Angles are of different Species.

WHEN the Obtuse Value of DA is less than BA, BD will be had by substracting either Value of DA from BA; and then BD is Ambiguous. But when the Obtuse Value of DA *is not* less than BA, the Acute Value of DA, taken from BA leaves the only Value of BD; which therefore is not Ambiguous; tho' in both Varieties the Perpendicular fell without.

In the third, we have the same Omission; where there are given two Sides BC, CD, *and* B *an Angle opposite to* CD *one of them, to find the third Side* BD.

FIRST, we may observe, that the Species of DA is always known; for it is of $\left\{\begin{matrix}\text{the same}\\\text{a different}\end{matrix}\right\}$ Affection with the Angle B, when DC is $\left\{\begin{matrix}\text{less}\\\text{greater}\end{matrix}\right\}$ than a Quadrant. And,

If AD be less than AB, and also the Sum of AD and AB less than a Semicircle; then AD, either added to, or substracted from AB, will give the Value of BD, which, therefore, is Ambiguous.

But if AD be not less than AB. or if their Sum be be not less than a Semicircle; then their Sum in the former, and their Difference in the latter Variety, shall give one single Value of BD, and then is not Ambiguous.

The seventh Case much resembles the third; for there are given two Sides BC, CD, *and* B *an Angle, opposite to* CD *one of them; to find the Angle* BCD, *lying between those two Sides.*

AND here we may observe, that the Species of the Angle DCA is known; for it is of $\left\{\begin{array}{l}\text{the same}\\ \text{a different}\end{array}\right\}$ kind with the Angle B, when DC is $\left\{\begin{array}{l}\text{less}\\ \text{greater}\end{array}\right\}$ than a Quadrant. And,

If DCA be less than BCA, and the Sum of DCA and BCA less than 2 Right Angles; then, DCA either added to, or substracted from BCA will give the Angle BCD; which therefore is Ambiguous.

If DCA be not less than BCA, or the Sum of DCA and BCA not less than 2 Right Angles; then their Sum in the former, and their Difference in the latter Variety shall give the Single Value of BCD; which then is not Ambiguous.

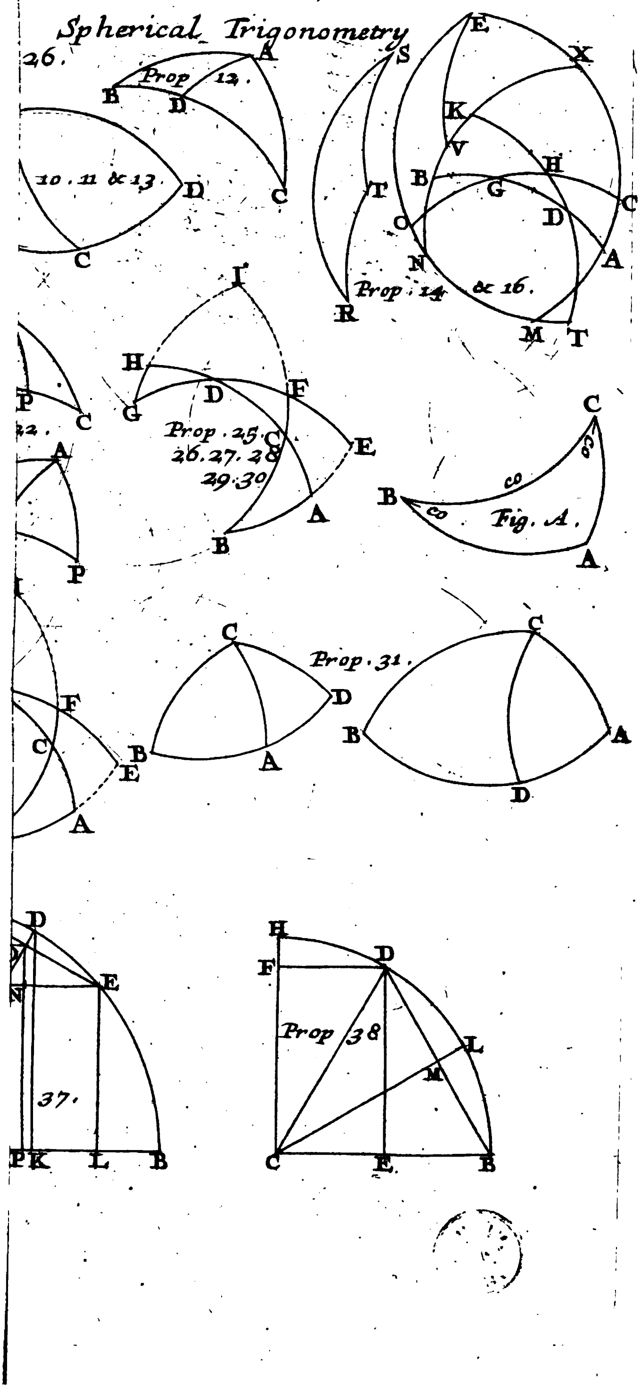
N.B. If any one will be at the Trouble to make a double Calculation for the Side DC, or the Angle D, as taught in the Remarks on the 9th and 10th Cases, they will find the several Varieties in the 1st, 3d, 5th, and 7th, to be as here laid down in these easy Rules.

The Truth of these Rules may be easily deduced from the 10*th*, 13*th*, 18*th*, and 22*d Prop. of this*, and the 2*d*, 8*th*, and 13*th Examples*, following *Prop.* 30, *of this.*

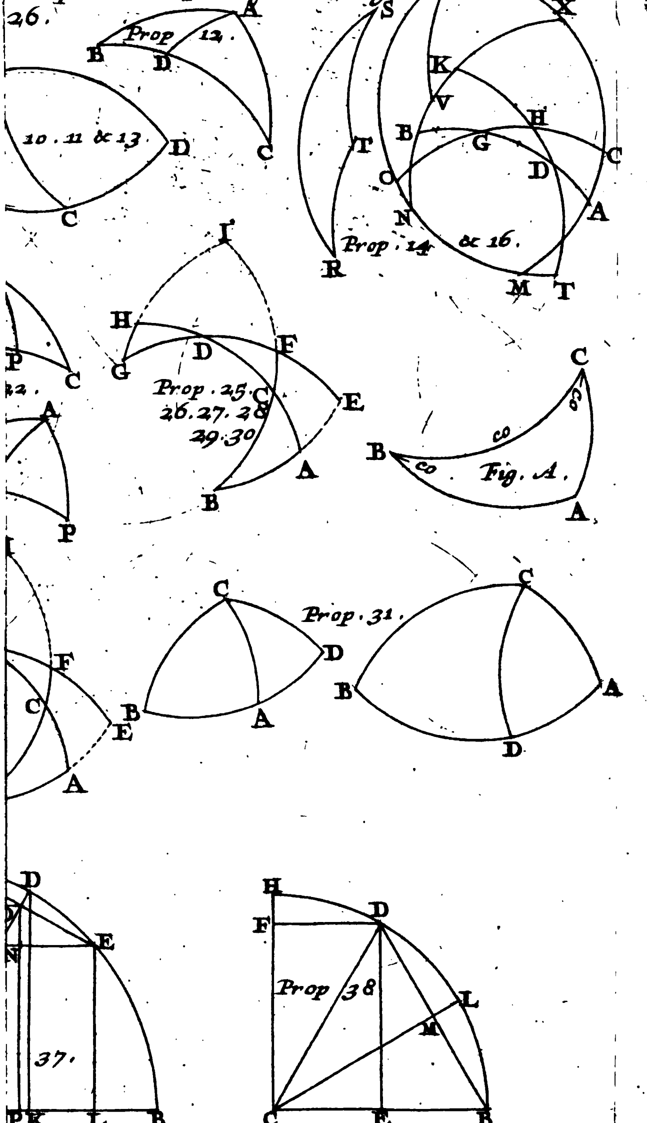
In our third *Case* of Oblique-plain Triangles, our Author should have added this.

If AB be less than BC, the Angle A is Ambiguous, otherwise not.

Spherical Trigonometry



Spherical Trigonometry



A Short

TREATISE

OF THE

NATURE and ARITHMETICK

OF

LOGARITHMS.

THE

PREFACE.

THE Mathematicks formerly received considerable Advantages; first, by the Introduction of the Indian *Characters, and afterwards by the Invention of Decimal Fractions; yet has it since reaped at least as much from the Invention of Logarithms, as from both the other two. The Use of these, every one knows, is of the greatest Extent, and runs through all Parts of Mathematicks. By their Means it is that Numbers almost infinite, and such as are otherwise impracticable, are managed with Ease and Expedition. By their Assistance the Mariner steers his Vessel, the Geometrician investigates the Nature of the higher Curves, the Astronomer*

mer determines the Places of the Stars, the Philosopher accounts for other Phenomena of Nature; and lastly, the Usurer computes the Interest of his Money.

The Subject of the following Treatise has been cultivated by Mathematicians of the first Rank; some of whom taking in the whole Doctrine, have indeed wrote learnedly, but scarcely intelligible to any but Masters. Others, again, accommodating themselves to the Apprehension of Novices, have selected out some of the most easy and obvious Properties of Logarithms, but have left their Nature and more intimate Properties untouch'd. My Design therefore in the following Tract, is to supply what seemed still wanting, viz. *to discover and explain the Doctrine of Logarithms, to those who are not yet got beyond the Elements of Algebra and Geometry.*

The wonderful Invention of Logarithms we owe to the Lord Neper, *who was the first that constructed and published a Canon thereof, at* Edenburgh, *in the Year* 1614. *This was very graciously received by all Mathematicians, who were immediately sensible of the extreme Usefulness thereof. And tho' it is usual to have various Nations contending for the Glory of any notable Invention, yet* Neper *is universally allow'd the Inventor of Logarithms, and enjoys the whole Honour thereof without any Rival.*

The same Lord Neper *afterwards invented another and more commodious Form of Logarithms, which he communicated to Mr.* Henry Briggs, *Professor of Geometry at* Oxford, *who was hereby introduced as a Sharer in the compleating thereof: But the Lord* Neper *dying, the whole Business remaining, was devolved upon Mr.* Briggs, *who, with prodigious Application, and an uncommon Dexterity, compos'd a Logarithmick Canon, agreeable to that new Form for the first twenty Chiliads of Numbers, (or from* 1 *to* 20000) *and for eleven other Chiliads,* viz. *from* 90000 *to* 101000. *For all which Numbers he calculated the Logarithms to fourteen Places of Figures. This Canon was publish'd at* London *in the Year* 1624.

Adrian Vlacq *published again this Canon at* Goudæ *in* Holland, *in the Year* 1628, *with the intermediate Chiliads before omitted, filled up according to* Brigg's *Prescriptions; but these Tables are not so useful as* Brigg's, *because the Logarithms are continued but to* 10 *Places of Figures.*

Mr. Briggs *also has calculated the Logarithms of the Sines and Tangents of every Degree, and the hundredth Parts of Degrees to* 15 *Places of Figures, and has subjoined to them the Natural Sines, Tangents, and Secants, to* 15 *Places of Figures. The Logarithms of the Sines*

and Tangents are called Artificial Sines and Tangents. These Tables, together with their Construction and Use, was publish'd after Brigg's *Death, at* London, *in the Year* 1633, *by* Henry Gellibrand, *and by him called,* Trigonometria Britannica.

Since then there have been published, in several Places, compendious Tables, wherein the Sines and Tangents, and their Logarithms, consist of but seven Places of Figures, and wherein are only the Logarithms of the Numbers from 1 *to* 100000, *which may be sufficient for most Uses.*

The best Disposition of these Tables, in my Opinion, is that first Thought of by Nathaniel Roe, *of* Suffolk; *and with some Alterations for the better, followed by* Sherwin *in his Mathematical Tables, publish'd at* London *in* 1705; *wherein are the Logarithms from* 1 *to* 101000, *consisting of* 7 *Places of Figures. To which are subjoined the Differences and proportional Parts, by Means of which may be found easily the Logarithms of Numbers to* 10000000, *observing at the same Time that these Logarithms consist only of* 7 *Places of Figures. Here are also the Sines, Tangents, and Secants, with the Logarithms and Differences for every Degree and Minute of the Quadrant, with some other Tables of Use in practical Mathematicks.*

OF

OF THE

NATURE and ARITHMETICK OF LOGARITHMS.

CHAP. I.

Of the ORIGIN *and* NATURE *of Logarithms.*

AS in Geometry, the Magnitudes of Lines are often defined by Numbers; so likewise on the other hand, it is sometimes expedient to expound Numbers by Lines, *viz.* by assuming some Line which may represent Unity, and the Double thereof; the Number 2, the Triple 3, the one half, the Fraction $\frac{1}{2}$, and so on. And thus the Genesis and Properties of some certain Numbers are better conceived, and more clearly considered, than can be done by abstract Numbers.

Hence, if any Line *a* * be drawn into itself, the Quantity *aa* produced thereby, is not to be taken as one of two Dimensions, or as a Geometrical Square, whose Side is the Line *a*, but as a Line that is a third Proportional * Fig. 1.

portional to some Line taken for Unity and the Line a. So likewise, if a^2 be multiplied by a, the Product a^3, will not be a Quantity of three Dimensions, or a Geometrical Cube, but a Line that is the fourth Term in a Geometrical Progression, whose first Term is 1, and second a; for the Terms 1, a, a^2, a^3, a^4, a^5, a^6, a^7, &c. are in the continual Ratio of 1 to a. And the Indices affixed to the Terms, shew the Place or Distance that every Term is from Unity. For Example, a^5 is in the fifth Place from Uuity, a^6 is the sixth, or six times more Distant from Unity than a, or a^1, which immediately follows Unity.

If between the Terms 1 and a, there be put a mean Proportional which is $\sqrt{a}$, the Index of this will be $\frac{1}{2}$, for its Distance from Unity will be one half of the Distance of a from Unity; and so $a^{\frac{1}{2}}$ may be written $\sqrt{a}$. And if a mean Proportional be put between a and a^2, the Index thereof will be $1\frac{1}{2}$ or $\frac{3}{2}$, for its Distance will be sesquialteral of the Distance of a from Unity.

If there be two mean Proportionals put between 1 and a; the first of them is the Cube Root of a, whose Index must be $\frac{1}{3}$, for that Term is distant from Unity only by a third Part of the Distance of a from Unity; and so the Cube Root must be expressed by $a^{\frac{1}{3}}$. Hence, the Index of Unity is 0. for Unity is not distant from itself.

The same Series of Quantities, Geometrically Proportional, may be both ways continued, as well descending towards the Left Hand, as ascending towards the Right; for the Terms $\frac{1}{a^5}$, $\frac{1}{a^4}$, $\frac{1}{a^3}$, $\frac{1}{a^2}$, $\frac{1}{a}$, 1, a, a^2, a^3, a^4, a^5, &c. are all in the same Geometrical Progression. And since the Distance of a from Unity is towards the Right Hand, and positive or $+1$, the Distance equal to that on the contrary Side, *viz.* the Distance of the Term $\frac{1}{a}$ will be Negative or -1, which shall be the Index of the Term $\frac{1}{a}$, for which may be written a^{-1}. So likewise in the Term a^{-2}. The Index -2 shews that that Term stands in the second Place from Unity towards the Left Hand, and the

the Terms a^{-2} and $\frac{1}{a^2}$ are of the ſame Value. Alſo a^{-3} is the ſame as $\frac{1}{a^3}$ For theſe negative Indices ſhew that the Terms belonging to them, go from Unity the contrary way to that by which the Terms whoſe Indices are poſitive do. Theſe Things premiſed.

If on the Line A N, both ways indefinitely extended, be taken, A C, C E, E G, G I, I L, on the right Hand. And alſo A Γ, Γ Π, *&c.* on the left, all equal to one another. And if at the Points Π, Γ, A, C, E, G, I, L, be erected to the Right Line A N, the Perpendiculars Π Σ, Γ Δ, A B, C D, E F, G H, I K, L M, which let be continually proportional, and repreſent Numbers, whereof A B is Unity. The Lines A C, A E, A G, A I, A L—A Γ,—A Π, reſpectively expreſs the Diſtances of the Numbers from Unity, or the Place and Order that every Number obtains in the Series of Geometrical Proportionals, according as it is diſtant from Unity. So ſince A G is triple of the Right Line A C, the Number G H ſhall be in the third Place from Unity, if C D be in the firſt: So likewiſe ſhall L M be in the fifth Place, ſince A L = 5 A C. If the Extremities of the Proportionals Σ, Δ, B, D, F, H, K, M, be joined by Right Lines, the Figure Σ Δ L M will be become a Polygon conſiſting of more or leſs Sides, according as there are more or leſs Terms in the Progreſſion.

If the Parts A C, C E, E G, G I, I L, be biſected in the Points *c*, *e*, *g*, *i*, *l*, and there be again raiſed the Perpendiculars *cd*, *ef*, *gh*, *ik*, *lm*, which are mean Proportionals between A B, C D; C D, E F; E F, G H; G H, I K; I K, L M; then there will ariſe a new Series of Proportionals, whoſe Terms beginning from that which immediately follows Unity, are double of thoſe in the firſt Series, and the Difference of the Terms are become leſs, and approach nearer to a Ratio of Equality than before. Likewiſe in this new Series, the Right Lines A L, A C, expreſs the Diſtances of the Terms L M, C D, from Unity, *viz.* Since A L is ten times greater than A *c*, L M ſhall be the tenth Term of the Series from Unity: And becauſe

A *c*

A e is three times greater than A c, $e f$ will be the third Term of the Series, if $c d$ be the first; and there shall be two mean Proportionals between A B and $e f$, and between A B, and L M, there will be nine mean Proportionals.

And if the Extremities of the Lines B d D f F h H, &c. be joined by Right Lines, there will be a new Polygon made, consisting of more, but shorter Sides than the last.

If, again, the Distances A c, c C, C e, e E, &c. be supposed to be bisected, and mean Proportionals between every two of the Terms, be conceived to be put at those middle Distances; then there will arise another Series of Proportionals, containing double the Number of Terms from Unity than the former does; but the Differences of the Terms will be less; and if the Extremities of the Terms be joined, the Number of the Sides of the Polygon will be augmented according to the Number of Terms; and the Sides thereof will be lesser, because of the Diminution of the Distances of the Terms from each other.

Now in this new Series, the Distances A L, A C, &c. will determine the Orders or Places of the Terms; *viz.* if A L be five times greater than A C, and C D be the fourth Term of the Series from Unity, then L M will be the twentieth Term from Unity.

If in this Manner mean Proportionals be continually placed between every two Terms, the Number of Terms at last will be made so great, as also the Number of the Sides of the Polygon, as to be greater than any given Number, or to be infinite; and every Side of the Polygon so lessened, as to become less than any given Right Line; and consequently the Polygon will be changed into a Curve-lin'd Figure; for any Curve-lin'd Figure may be conceived as a Polygon, whose Sides are infinitely small and infinite in Number.

A Curve described after this Manner, is called *Logarithmical*; in which, if Numbers be represented by Right Lines standing at Right Angles to the Axis A N, the Portion of the Axis intercepted between any Number and Unity, shews the Place or Order that that Number obtains in the Series of Geometrical Proportionals, distant from each other by equal Intervals.

For

For Example, if AL be five times greater than AC, and there are a thousand Terms in continual Proportion from Unity to LM; then will there be two Hundred Terms of the same Series from Unity to CD, or CD shall be the two hundredth Term of the Series from Unity; and let the Number of Terms from AB to LM be supposed what it will; then the Number of Terms from AB to CD, will be one fifth Part of that Number.

The Logarithmical Curve may also be conceived to be described by two Motions, one of which is equable, and the other accelerated or retarded according to a given Ratio. For Example, if the Right Line AB, moves uniformly along the Line AN, so that the End A therefore describes equal Spaces in equal Times; and, in the mean time, the said Line AB so increases, that the Increments thereof generated in equal Times, be proportional to the whole increasing Line, that is, if AB in going forward to *cd*, be encreased by the Increment *od*, and in an equal Time when it is come to CD, the Increment thereof is D*p*, and D*p* to *dc*, is as *do* is to AB, that is, if the Increments generated in equal Times are always proportional to the Wholes; or, if the Line AB moving the contrary Way, diminishes in a constant Ratio, so that while it goes thro' the equal Spaces, the Decrements AB—ΓΔ, ΓΔ—ΠΣ, are Proportionals to AB, ΓΔ. Then the End of the Line increasing or decreasing in the said Manner, describes the Logarithmcal Curve: For since AB : *do* :: *dc* : D*p* :: DC : *fq*, it shall be (by Composition of Ratio) as AB : *dc* :: *de* : DC :: DC : *fe*. and so on.

By these two Motions, *viz*. the one equable, and the other proportionally accelerated or retarded, the Lord *Neper* laid down the Origin of Logarithms, and call'd the Logarithm of the Sine of any Arc, *That Number which nearest defines a Line that equally encreases, while, in the mean time, the Line expressing the whole Sine proportionally decreases to that Sine*.

It is manifest from this Description of the Logarithmick Curve, that all Numbers at equal Distances are continually proportional. It is also plain, that if there be four Numbers AB, CD, IK, LM, such, that the Dis-

tance between the first and second, be equal to the Distance between the third and the fourth: Let the Distance from the second to the third be what it will, these Numbers will be proportional. For because the Distances AC, IL, are equal, AB shall be to the Increment D *s*, as IK is to the Increment MT. Wherefore (by Composition) AB : DC : : IK : ML. And contrariwise, if four Numbers be proportional, the Distance between the first and the second, shall be equal to the Distance between the third and the fourth.

The Distance between any two Numbers, is called the Logarithm of the Ratio of those Numbers, and indeed doth not measure the Ratio itself, but the Number of Terms in a given Series of Geometrical Proportionals proceeding from one Number to another, and defines the Number of equal Ratio's by the Composition whereof the Ratio of Numbers are known.

If the Distance between any two Numbers be double to the Distance between two other Numbers, then the Ratio of the two former Numbers shall be the Duplicate of the Ratio of the two latter. For let the Distance IL between the Numbers IK, LM, be double to the Distance A*c*, between the Numbers AB, *cd*; and since IL is bisected in *l*, we have A*c* $=$ I*l* $=$ *l*L; and the Ratio of IK to *lm*, is equal to the Ratio of AB to *cd*; and so the Ratio of IK to LM, the Duplicate of the Ratio of IK to *lm*, (by *Def.* 10. *El.* 5.) shall be the Duplicate of the Ratio of AB to *cd*.

In like Manner, if the Distance EL be triple of the Distance AC, then will the Ratio of EF to LM, be triplicate of the Ratio of AB to CD: For because the Distance is triple, there shall be three times more Proportionals from EF to LM, than there are Terms, of the same Ratio, from AB to CD; and the Ratio of EF to LM, as also of AB to CD, is compounded of the equal intermediate Ratio's, (by *Def.* 5. *El.* 6.) And so the Ratio of EF to LM, compounded of three times a greater Number of Ratio's, shall be triplicate of the Ratio of AB to CD. So likewise if the Distance GL be quadruple of the Distance A*c*, then shall the Ratio of GH to LM, be quadruplicate of the Ratio of AB to *cd*.

The

The Logarithm of any Number, is the Logarithm of the Ratio of Unity to that Number, or it is the Distance between Unity and that Number. And so Logarithms express the Power, Place, or Order which every Number, in a Series of Geometrical Progressionals, obtains from Unity. For Example, if there be 10000000 proportional Numbers from Unity to the Number 10, that is, if the Number 10 be in the 10000000th Place from Unity; then it will be found, by Computation, that in the same Series from Unity, to 2 there are 3010300 proportional Terms, that is, the Number 2 will stand in the 3010300th Place. In like Manner, from Unity to 3, there will be found 4771213 proportional Terms, which Number defines the Place of the Number 3. The Numbers 10000000, 3010300, 4771213, shall be the Logarithms of the Numbers 10, 2, and 3.

If the first Term of the Series from Unity be called y, the second Term will be y^2, the third y^3. &c. And since the Number 10 is the 10,000,000th Term of the Series, then will $y^{10000000}=10$. Also $y^{3010300}=2$. Also $y^{4771213}=3$; and so on.

Wherefore all Numbers shall be some Powers of that Number which is the first from Unity; and the Indices of the Powers are the Logarithms of the Numbers.

Since Logarithms are the Distances of Numbers from Unity, as has been shewn. The Logarithm of Unity shall be 0, for Unity is not distant from itself. but the Logarithms of Fractions are negative, or descending below nothing, for they go on the contrary Way. And so if Numbers increasing proportionally from Unity, have positive Logarithms, or such as are affected with the Sign +; then Fractions or Numbers in like Manner decreasing, will have negative Logarithms, or such as are affected with the Sign —; which is true when Logarithms are considered as the Distances of Numbers from Unity.

But if Logarithms take their Beginning not from an integral Unite, but from a Unite that is in some Place of Decimal Fractions. For Example, from the Fraction $\frac{1}{10\,00000\,0000}$; then all Fractions greater than this, will have positive Logarithms, and those that are less,

will have negative Logarithms. But more shall be said of this hereafter.

Since in the Numbers continually proportional, D C, E F, G H, I K, &c. the Distances C E, E G, G I, &c. are equal, the Logarithms A C, A E, A G, A I, &c. of those Numbers shall be equidifferent, or the Differences of them shall be equal: And so the Logarithms of proportional Numbers are all in an arithmetical Progression; and from hence proceeds that common Definition of Logarithms, that Logarithms are Numbers which, being adjoined to Proportions, have equal Differences.

In the first Kind of Logarithms that *Neper* published, the first Term of the continual Proportionals, was placed only so far distant from Unity, as that Term exceeded Unity. For Example, if $v\,n$ be the first Term of the Series from Unity A B, the Logarithm thereof, or the Distance A n, or B y, was, according to him, equal to $v\,y$, or the Increment of the Number above Unity. As suppose $v\,n$ be 1,0000001, he placed 0,0000001 for its Logarithm A n; and from hence, by Computation, the Number 10 shall be the 23025850th Term of the Series, which Number therefore is the Logarithm of 10 in this Form of Logarithms, and expresses its Distance from Unity in such Parts whereof $v\,y$ or A n is one.

But this Position is entirely at Pleasure, for the Distance of the first Term may have any given Ratio to the Excess thereof above Unity, and according to that various Ratio (which may be supposed at Pleasure,) that is between $v\,y$ and B y, the Increment of the first Term above Unity, and the Distance of the same from Unity, there will be produced different Forms of Logarithms.

This first Kind of Logarithms was afterwards changed by *Neper*, into another more convenient one wherein he put the Number 10 not as the 23025850th Term of the Series, but the 1000,0000th; and in this Form of Logarithms, the first Increment $v\,y$ shall be to the Distance B y, or A n, as Unity, or A B, is to the Decimal Fraction 0,4342994, which therefore expresses the Length of the Subtangent A T. *Fig.* 4.

After *Neper*'s Death, the excellent Mr. *Henry Briggs*, by great Pains, made and published Tables of Logarithms

rithms according to this Form. Now since in these Tables, the Logarithm of 10, or the Distance thereof from Unity, is 1.0000000, and 1,10,100,1000, 10000, &c. are continual Proportionals, they shall be equidistant. Wherefore the Logarithm of the Number 100 shall be 2,0000000; of 1000, 3,0000000; and the Logarithm of 10000 shall be 4,0000000; and so on.

Hence the Logarithms of all Numbers between 1 and 10, must begin with 0, or 0 must stand in the first Place to the Left-Hand; for they are lesser than the Logarithm of the Number 10, whose Beginning is Unity; and the Logarithms of the Numbers between 10 and 100 begin with Unity; for they are greater than 1,0000000, and less than 2,0000000. Also the Logarithms between 100 and 1000, begin with 2, for they are greater than the Logarithm of 100, which begins with 2, and less than the Logarithm of a 1000 that begins with 3. In the same Manner it is demonstrated, that the first Figure to the Left-Hand of the Logarithms between 1000 and 10000, must be 3; and the first Figure to the Left-Hand of the Logarithms between 10000 and 100000, will be 4; and so on.

The first Figure of every Logarithm to the Left-Hand, is called the Characteristick or Index, because it shews the highest or most remote Place of the Number from the Place of Unites. For Example, if the Index of a Logarithm be 1, then the highest or most remote Place from Unity of the correspondent Number to the Left-Hand, will be the Place of Tens. If the Index be 2, the most remote Figure of the correspondent Number shall be in the second Place from Unity, that is, it shall be in the Place of Hundredths; and if the Index of a Logarithm be 3, the last Figure of the Number answering to it, shall be in the Place of Thousandths. The Logarithms of all Numbers that are in Decuple or Subdecuple Progression, only differ in their Characteristicks, or Indices, they being written in all other Places with the same Figures. For Example, the Logarithms of the Numbers 17, 170, 1700, 17000, are the same, unless in their Indices; for since 1 is to 17, as 10 to 170, and as 100 to 1700, and as 1000 to 17000; therefore the Distances be-

tween 1 and 17, between 10 and 170, between 100 and 1700, and between 1000 and 17000, ſhall be all equal. And, ſo ſince the Diſtance between 1 and 17, or the Logarithm of the Number 17 is 1.2304489, the Logarithm of the Number 170, will be=2.2304489, and the Logarithm of the Number 1700 ſhall be 3.2304489, becauſe the Logarithm of the Number 100=2.0000000. In like Manner, ſince the Logarithm of the Number 1000=3.0000000, the Logarithm of the Number 17000 ſhall 4.2304489.

So alſo the Numbers, 6748. 674, 8. 67, 48. 6, 748. 0, 6748. 0, 06748, are continual Proportionals in the Ratio of 10 to 1; and ſo their Diſtances from each other ſhall be equal to the Diſtance or Logarithm of the Number 10, or equal to 1,0000000. And ſo ſince the Logarithm of the Number 6748 is 3,8291751, the Logarithms of the other Numbers ſhall be as in the Margin; where you may obſerve that the Indices of the laſt two Logarithms are only negative, and the other Figures poſitive; and ſo when thoſe other Figures are to be added, the Indices muſt be ſubſtracted, and contrariwiſe.

6748	3,8291751
674,8	2,8291751
67,48	1,8291751
6,748	0,8291751
0,6748	—1,8292751
0,06748	—2,8291751

CHAP.

CHAP. II.

Of the Arithmetick of Logarithms in whole Numbers, or whole Numbers adjoined to Decimal Fractions. Fig. 2.

BECAUSE, in Multiplication, Unity is to the Multiplier, as the Multiplicand is to the Product, the Distance between Unity and the Multiplier, shall be equal to the Distance between the Multiplicand and the Product; if therefore, the Number GH be to be multiplied by the Number EF, the Distance between GH and the Product must be equal to the Distance AE, or to the Logarithm of the Multiplier; and so if GL be taken equal to AE, the Number LM shall be the Product, that is, if the Logarithm of the Multiplicand AG be added, the Logarithm of the Multiplier AE, the Sum shall be the Logarithm of the Product.

In Division, Unity is to the Divisor, as the Dividend is to the Quotient; and so the Distance between the Divisor and Unity shall be equal to the Distance between the Dividend and the Quotient. So if LM be to be divided by EF, the Distance EA shall be equal to the Distance between LM and the Quotient, and so if LG be taken equal to EA, the Quotient will be at G; that is, if from AL, the Logarithm of the Dividend, be taken GL, or AE, the Logarithm of the Divisor, there will remain AG, the Logarithm of the Quotient.

And from hence it appears, that whatsoever Operations in common Arithmetick are performed by multiplying or dividing of great Numbers, may be much easier, and more expediently done by the Addition or Substraction of Logarithms.

Let, for Example, the Number 7589 be to be multipled by 6757. Now, if the Logarithms of those Numbers be added together, as in the Margine, their Sum will be the Logarithm of the Product, whose Index 7

Log. 3.	8801846
Log. 3.	8297539
Log. 7.	7099385.

shews that there are seven Places of Figures, besides Unity, in the Product; and in seeking this Logarithm in Tables, or the nearest equal to it, I find that the Number answering thereto, which is lesser than the Product is 51278000, and the Number greater than the Product is 51279000, and if the adjoined Differences and proportional Parts be taken, the Numbers that must be added to the Place of Hundreds and Tens in the Product are 87, and that which must be added in the Place of Unity, will necessarily be 3, since seven times nine = 63, and so the true Product shall be 51278873. If the Index of the Logarithm had been 8 or 9, then the Numbers to be added in the Place of Hundredths or Tenths, could not be had from those Tables of Logarithms which consist of but 7 Places of Figures, besides the Characteristick, and so in this Case, the *Ulacquian* or *Briggian* Tables should be used; in the former of which, the Logarithms are all to ten Places of Figures, and in the latter to fourteen.

If the Number 78956 be to be divided by 278, by substracting the Logarithm of the Divisor from the Logarithm of the Dividend, the Logarithm of the Quotient will be had. And to this Logarithm, the Number 282, 719 answers; which therefore shall be the Quotient.

Log. 4.	8954004
Log. 2.	4440448
Log. 2.	4513556

Because Unity, any assumed Number, the Square thereof, the Cube, the Biquadrate, &c. are all continual Proportionals, their Distances from each other shall be equal to one another. And so it is manifest, that the Distance of the Square from Unity, is double of the Distance of its Root from the same: Also the Distance of the Cube, is triple of the Distance of its Root; and the Distance of the Biquadrate, is quadruple of the Distance of Its Root from Unity, &c. And so if the Logarithm of any Number be doubled, we shall have the Logarithm of its Square; if it be tripled, we shall have the Logathim of its Cube, and if it be

qua-

quadrupled, the Logarithm of its Biquadrate. And contrariwise, if the Logarithm of any Number be bisected, we shall have the Logarithm of the Square Root thereof: Moreover, a third Part of the said Logarithm, will be the Logarithm of the Cube Root of the Number; and a fourth Part, the Logarithm of the Biquadrate Root of that Number.

Hence, the Extraction of all Roots are easily performed, by dividing a Logarithm into as many Parts as there are Units in the Index of the Power. So if you want the Square Root of 5, the half of 0,6989700 must be taken, and then that half 0.3494850 will be the Logarithm of the Square Root of 5, or the Logarithm of $\sqrt{5}$, to which the Number 2.23606 nearly answers.

CHAP. III.

Of the Arithmetick of Logarithms, when the Numbers are Fractions. Fig. 3.

WHEN Fractions are to be worked by Logarithms, it is necessary, for avoiding the Trouble of adding one Part of a Logarithm, and substracting the other, that Logarithms do not begin from an integral Unit, but from some Unit that is the tenth or hundredth Place of Decimal Fractions: For Example, let PO be $\frac{1}{10000000000}$ and from this let the Logarithms begin. Now this Fraction is ten times more distant from Unity to the Left Hand, than the Number 10 is distant therefrom to the Right; for there are 10 proportional Terms in the Ratio of 10 to 1, from Unity to PO. And so if AB be Unity, the Logarithm thereof, according to this Supposition, will not be 0, but OA will be = 10.0000000; for the Distance of any Tenth from Unity is 1.0000000, whence the Distance of the Number 10 from

from PO will be 11. 0000000. Also the Distance of the Number 100 from PO, or its Logarithm, beginning from PO, shall be 12.000 0000, and the Logarithm of 1000, or the Distance from PO, will be 13.000 0000. And thus, the Indices of all Logarithms are augmented by the Number 10; and those Fractions whose Indices are —1, or —2, or —3, *&c.* are now made 9, 8, or 7, *&c.*

But if Logarithms begin from the Place of a Fraction, whose Numerator is Unity, and Denominator Unity with 100 Cyphers added to it, (which they must do when Fractions occur that are less than PO) than that Fraction will be 100 times more *distant* from Unity, than 10 is distant from it; and so the Logarithm of Unity will have 100 for the Index thereof. And the Logarithm of any Tens will have 101 for the Index, that of any Hundreds 102, and so on; all the Indices being augmented by the Number 100.

The Logarithms of all Fractions that are greater than PO (whereat they begin) will be positive. And since the the Numbers 10, 1, $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, *&c.* are in a continued Geometrical Progression, they will be equally distant from each other; and accordingly their Logarithms will be equidifferent: And so when the Logarithm of 10, is 11.0000000, and the Logarithm of Unity is 10.0000000, and the Logarithm of the Fraction $\frac{1}{10}$ will be 9.0000000, and the Logarithm of the Fraction $\frac{1}{100}$ will be 8. 0000000, and in like manner, the Index of the Logarithm of $\frac{1}{1000}$ will be 7. Also for the same Reason, if the Index of the Logarithm of Unity be 100, and of 10 be 101, then will the Index of the Logarithm of the Fraction $\frac{1}{10}$ be 99, and the Index of the Logarithm of $\frac{1}{100}$ will be 98, and the Index of Logarithm of the Fraction $\frac{1}{1000}$ shall be 97, *&c.* And these Indices shew in what Place from Unity, the first Figure of the Fraction, not being a Cypher, must be put. For Example, if the Index be 4, the Distance thereof from the Index of Unity, (which is 10) *viz.* 6, shews that the first significative Figure of the Decimal, is in the sixth Place from Unity; and therefore, five Cyphers are to be prefixed thereto towards the Left Hand. So also if the Index of Unity be 100, and the Index of the Fraction be 80, the

the first Figure thereof shall be in the 20th Place from Unity, and 19 Cyphers are to be prefixed thereto.

Now, let it be required to multiply the Fraction GH by the Fraction DC. Because Unity is to the Multiplier, as the Multiplicand is the Product; the Distance between Unity and the Multiplier shall be equal to the Distance between the Multiplicand and the Product. Therefore, if there be taken GI=AC, the Product IK shall be at I. And accordingly, if from OG, the Logarithm of the Multiplicand, there be taken GI or AC, there will remain OI, the Logarithm of the Product. But AC=OA—OC, which taken from OG, there will remain OG+OC—OA=OI, that is, if the Logarithm of the Multiplier and Multiplicand be added together, and from the Sum be taken the Logarithm of Unity, (which is always expressed by 10 or 100 with Cyphers) the Logarithm of the Product will be had. For Example, let the Decimal Fraction 0, 00734 be to be multiplied by the Fraction 0, 000876. Set down 100 for the Index of the Logarithm of Unity, and then the Logarithms of the Fractions will be as in the Margine, which being added together, and the Logarithm of Unity being taken away from the Sum, the Remainder is the Logarithm of the Product, whose Index 94 shews that the first Figure of the Product is in the sixth Place from Unity, and so there must be five Cyphers prefixed, and then the Product will be ,00000642984.

97,8656961
96,9425041
94.8082002

In Division, the Divisor is to Unity, as the Dividend is to the Quotient; and so the Distance between the Divisor and Unity shall be equal to the Distance between the Dividend and the Quotient. And so if the Fraction IK be to be divided by DC, you must take IG=CA, and the Place of the Quotient shall be G. But CA=OA—OC, which being added to OI, we have OA+OI—OC=OG, that is, if the Logarithm of Uuity be added to the Logarithm of the Dividend, and from the Sum be taken the Logarithm of the Divisor, there will remain the Logarithm of the Quotient; so if the Number CD be to be divided by IK, you must take the Distance CS=IA, and then ST will be the Quotient, whose Logarithm

rithm is OA+OC—OI. Let CD=0. 347, IK =0. 00478. Then add the Logarithm of Unity to the Logarithm of CD, that is, put 1 or 10 before the Index thereof, and from that substract the Logathim of the Divisor, and the Remainder will be the Logarithm of the Quotient, whose Index 11. shews that the Quotient is between the Numbers 10 and 100; and I seek the Number answering the Logarithm, which I find to be 72, 549. If the Logarithm of a Vulgar Fraction, for Example, $\frac{7}{8}$ be required, the Logarithm of Unity must be added to the Logarithm of the Numerator 7, or which is all one, you must put 10 or 100 before the Index thereof, and subduct from it the Logarithm of the Denominator 8, and there will remain the Logarithm of the Vulgar Fraction $\frac{7}{8}$, or the Decimal .875.

19. 5403295	10. 8450980
7. 6794279	0. 9030900
11. 8609016	9. 9420080

If the Powers of any Fraction DC be required, you must assume EC, EG, GI, IL, each equal to AC; and then EF will be the Square, GH the Cube, and IK the Biquadrate of the Number DC; for they are continually Proportional from Unity. Besides, AE= 2AC=2AO — 2OC, whence OE = OA—AE = 2OC—OA, that is, the Logarithm of the Square is the Double of the Logarithm of the Root, less the Logarithm of Unity. In like Manner, since AG= 3AC=3OA—3OC, we shall have OG=OA— AG=3OC—2OA= the Logarithm of the Cube =triplate the Logarithm of the Root, less the Double of the Logarithm of Unity For the same Reason, because AI=4AC=4OA—4OC, we have OI=4OC—3OA, which is the Logarithm of the Biquadrate. And universally, if the Power of a Fraction be n, and the Logarithm L, then shall the Logarithm of the Power n = nL—nOA+OA, that is, if the Logarithm of a Fraction be multiplied by n, and from the Product be taken the Logarithm of Unity, multiplied by n—1, the Logarithm of the Power n of that Fraction will be had.

For Example, it if is required to find the 6th Power of the Fraction $\frac{1}{20}$=, 05 the Logarithm of this Fraction is 8. 6989700, which being multiplied by 6, gives the Num-

Number 52. 1938200; and if from 52 the Number 50, which is the Index of the Logarithm of Unity drawn into 5, be taken away, the Remainder will be the Logarithm of the 6th Power, *viz.* 2. 1938200, to which the Number ,000000015625 answers. For the Index 2 shews that 7 Cyphers must be put before the first Figure.

If the 8th Power of the Fraction ,05 be required, by multiplying the Logarithm by 8, there will be produced 69. 5917600, and since 70, which is seven times the Index of the Logarithm of Unity, cannot be taken from 69, unless we run into negative Numbers, the Index of the Logarithm of Unity must be supposed 100, and then the Index of the Logarithm of the Fraction will be 98. Now this Logarithm drawn into 8 gives 789. 5917600, and if 700, which is 7 times the Index of the Logarithm of Unity be taken from 789, there will remain 89. 5917600, the Logarithm of the 8th Power of the Fraction $\frac{1}{20}$, whose correspondent Number is ,000000000039062, for since the Index is 89, and the Difference thereof from 100 is 11; the first significative Figure of the Fraction shall be in the 11th Place from Unity; and so there must 10 Cyphers be placed before it.

If the Roots of the Powers of Fractions be desired, for Example, the Square Root of the Fraction EF; because the Root is a mean Proportional between the Fraction and Unity, you must bisect AE in C, and then CD will be the square Root of the Fraction EF. But $AC = \frac{1}{2}AE = \frac{OA - OE}{2}$, and so the Logarithm of the Root $= OA - AC = \frac{OA + OE}{2}$. And if the Cube Root of the Fraction GH be sought, this shall be the first of two mean Proportionals between Unity and GH; and so if AG be divided into three equal Parts, the first of which is AC; then CD shall be the Root sought, and because $AC = \frac{1}{3}AG = \frac{OA - OG}{3}$, if this be taken from OA, there will remain $\frac{2OA + OG}{3} = OC =$ Logarithm of the Cube Root

Root of the Fraction GH. So likewise the Biquadrate Root of the Fraction IK will be had, by dividing AI into four equal Parts, for the Root is the first of three mean Proportinals between Unity and the Fraction, and consequently if $AC = \frac{1}{4} AI$, then will CD be the Biquadrate Root of the Fraction IK. But $\frac{1}{4} AI = \frac{OA - OI}{4}$ and so $OC = OA - AC = \frac{3OA + OI}{4}$.

And universally, if the Root of any Power n of the Fraction LM be required, the Logarithm of the Root thereof will be $\frac{nOA - OA + OL}{n}$, that is, if the Number $n - 1$ be perfix'd to the Index of the Logarithm, and the Logarithm thus augmented be divided by n, the Quotient will give the Logarithm of the Root sought. So if the Cube Root of the Fraction $\frac{1}{2}$ or .5 be sought, you must place $2 = n - 1$ (since the cube Root is required) before the Logarithm thereof, and there will be had 29, 6989700, a third Part of which is 9, 8996566, which is equal to the Logarithm of the Cube Root of the Fraction $\frac{1}{2}$, and the Number ,7937 answering to this Logarithm, is the Root sought.

CHAP. IV.

Of the Rule and Proportion by Logarithms.

THE Rule of Proportion shews how, by having three Numbers given, a fourth Proportional to them may be found, *viz.* if the second and third Terms be multiplied by one another, and the Product divided by the first Term, then will the Quotient be the fourth Proportional Term sought. But this fourth Term is much easier found by

by Logarithms; for if the Logarithm of the first Term be taken from the Sum of the Logarithms of the second and third Term, the Number remaining will be the Logarithm of the fourth sought.

Or this may be done something easier yet, if instead of the Logarithm of the first Term be taken its Complement Arithmetical, or the Difference of that Logarithm, and the Number 10. 0000000, which is done by setting down the Difference between each Figure of the Logarithm, and the Figure 9: for then if that Arithmetical Complement be added to the Sum of the other two Logarithms, and if Unity, which is the first Figure to the Left Hand, be taken from the Sum, the Remainder will be the Logarithm of the fourth Term sought; and so by this way, Logarithms of the fourth Term is found by only one Addition of three Numbers. The Reason of this will be manifest from hence: Let there be three Numbers A, B, C, from which the first is to be taken from the Sum of the second and third. Now this may not only be done by the common Way, but likewise, if there be any other third Number E taken, aud from this there be taken A, there will remain E—A, and if the Numbers B, C, and E—A be all added together, and from their Sum be taken E, there will remain B+C—A. So if the Number 15 be to be taken from 23, take the Complement of the Number 15 to 100, which is 85, and add this Number to 23, and the Sum will be 108, from which 100 being taken, there remains the Nuumber 8.

$$\begin{array}{r} 85 \\ \underline{23} \\ 108 \end{array}$$

Hence follow some Trigonometrical Examples of the Rule of Proportion solv'd by Logarithms.

Let ABC be a Right-lined Triangle, wherein are given, the Angle A 36 Degrees 46′, the Angle B 98 Degrees 32′, and the Side BC 3478, the Side AC is required. Say (by *Case* 1. of *Plain Trig.*) as the Sine of the Angle A is to the Sine of the Angle B, so is BC to AC. And because the Logarithm Sine of the

Arith. Comp. S, A.	0.2228938
Log. Sin. B.	9.9951656
Log. BC.	3.5413296
Log. AC.	3.7593890

Angle A is the first Term of the Analogy, I substitute its Complement Arithmetical for the same, and add the

the Logarithm of BC, the Logarithm of S, B, and the said Complement all three together, and reject Unity, which is in the first Place to the Left Hand, and then the Logarithm of the Side AC will be given, and the Number answering thereto is 5706,306 equal to the Side sought AC.

Let there be a spherical Triangle ABC, in which are given all the Sides, *viz.* BC=30 Degrees, AB=24 Degrees 4', and AC=42 Degrees 8', the Angle B is required. Let BA be produced to M, so that BM=BC, then will AM the Difference of the Sides BC, BA, be equal to 5 Degrees 56'. Now (by *Case* 11. in Oblique-angled Spherical Triangles) say, as the Rectangle under the Sines of the Legs, is to the Square of Radius, so is the Rectangle under the Sines of the Arcs $\frac{AC+AM}{2}$, $\frac{AC-AM}{2}$ to the Square of the Sine of one half the Angle B.

But $\frac{AC+AM}{2}$=24 Degrees 2', and $\frac{AC-AM}{2}$ =18 Degrees 6'; and because the first Term of the Analogy is the Rectangle under the Sines of AB, BC, and second Term is the Square of Radius, the Sum of the Logarithm Sine of AB, BC, must be taken from double the Logarithm of Radius, and what remains must be added to the Sum of the Logarithm S, of $\frac{AC+AM}{2}$, & $\frac{AC-AM}{2}$, which is the same as if the Logarithm Sines of each of the Arcs AB, BC, were substracted from the Log. of Radius, or if the Complements Arithmetical of these Sines be taken, and the Complements and the said Sines be all added together; then shall the Sum be the Logarithm of the Square of the Sine of half the Angle B. And so the half of the Logarithm

Log. S, BC Comp. Arith.	0.3010299
Log. S, AB Comp. Arith.	0.3898364
Log. S, $\frac{AC+AM}{2}$	9.6098803
Log. S, $\frac{AC-AM}{2}$	9.4923083
2. Log. S, Angle B.	19.7930549

garithm 9.8965274 is the Logarithm Sine of half the Angle B = 51 Degrees 59′. 56″, and the Double of this Angle shall be 103 Degrees 59′, 52″ = B, which was sought.

CHAP. V.

Of the continual Increments of proportional Quantities, and how to find by Logarithms, any Term in a Series of Proportionals, either increasing or decreasing. Fig. 3.

IF any where in the Axis of the Logarithmetical Curve, there be taken any Number of equal Parts SV, VY, YQ, &c. and at the Points S, V, Y, Q, &c. be raised the Perpendiculars ST, VX, YZ, QΠ, &c. then from the Nature of the Curve shall all these Perpendiculars be continually proportional; and therefore also the continual Increments X*x*, Z*z*, Π*π*, shall be proportional to their Wholes: For since ST : VX :: VX : YZ :: YZ : QΠ, it shall be (by Division of Proportion) ST : X*x* :: VX : Z*z* :: YZ : Π*π*, and (by Composition of Proportion) VX : X*x* :: YZ : Z*z* :: QΠ : Π*π*. Hence, if X*x* be any part of any Right Line ST, then will Z*z* be the same Part of the Right Line VX, and also Π*π*, the same Part of the Right Line YZ. For Example; if X*x* be the $\frac{1}{20}$ Part of ST, then will Z*z* = $\frac{1}{20}$ VX, and Π*π* = $\frac{1}{20}$ YZ; or which comes to the same, we shall have VX = ST + $\frac{1}{20}$ ST, YZ = VX + $\frac{1}{20}$ VX. Also QΠ = YZ + $\frac{1}{20}$ YZ.

Now make, as ST is to VX, so is Unity AB to NR; then shall AN = SV; and so each of the Right Lines SV, VY, YQ, &c. shall be equal to the Logarithm of RN, and AV, the Logarithm of the Term VX shall be equal to AS + AN = Logarithm of ST + Logarithm of NR. Also AY, the

Logarithm of the Term YZ, shall be equal to AS + 2AN = Logarithm ST + 2 Logarithm NR, and AQ, the Logarithm of the Term Qπ shall be equal to AS + 3AN = Logarithm ST + 3 Logarithm NR. And universally, if the Logarithm of the Number NR be multiplied by a Number, expressing the Distance of any Term from the first, and the Product be added to the Logarithm of the first Term, then will the Logarithm of that Term be had: But if a Series of Proportionals be decreasing, that is, if the Terms diminish in a continual Ratio, and Qπ be the first Term; then the Logarithm of any other will be had, in multiplying the Logarithm of the Number NR, by a Number that expresses the Distance of its Term from the first, and substracting the Product from the Logarithm of the first. And if the said Product be greater than the Logarithm of the first Term, then the Logarithms must begin from a Unit in some Place of Decimal Fractions, as from OP, and then the Logarithm of the Number Qπ will be OQ.

Now let LM represent any Money, or Sum of Money put out to Interest, so that the Interest thereof be accounted but at the End of every Year, and let K*k* be the Gain or Interest thereof at the End of the first Year, then will IK be the Sum of the Interest and Principal. And again, IK becoming the Principal at the End of the first Year, H*h* which is proportional to IK, or in a constant Ratio, will be the Gain at the end of the second Year; and so HG, at the End of the second Year, will become the Principal; and at the end of the third Year F*f*, proportional to GH, will be the Gain. Now let us suppose the Principal be augmented every Year $\frac{1}{20}$ Part thereof, so that IK = LM + $\frac{1}{20}$ LM, GH = IK + $\frac{1}{20}$ IK, EF = GH + $\frac{1}{20}$ GH, and so on. And accordingly, the Terms LM, IK, GH, EF, *&c.* continual Proportionals, it is required to find the Amount of the Money at the End of any Number of Years.

Let LM be a Farthing. Because LM is to IK, as 1 to 1 + $\frac{1}{20}$, or as 1 to 1.05, as AB is to NR, then will NR = 1.05, whose Logarithm AN. is 0.0211893, or more accurately 0.02118929991, it is required to find the Amount of a Farthing put out at compound Interest, at the End of 600 Years, multiply AN by

600

600, and the Product will be 12. 7135794, and to this Product add the Logarithm of the Fraction $\frac{1}{960}$, *viz.* 97.0177288 (for a Farthing is $\frac{1}{960}$ Part of a Pound) and the Sum 109. 7313082 shall be the Logarithm of the Number sought; and since the Index 109 exceeds the Index of Unity by 9, there shall be nine Places of Figures above Unity in the correspondent Number, and that Number being sought in the Tables, will be found greater than 5386500000, and less than 5386600000. And therefore a Farthing put out at Interest upon Interest, at 5 *per Cent. per Annum*, at the End of 600 Years will amount to above 5386500000 Pounds; which Sum could hardly be made up by all the Gold and Silver that has been dug out of the Bowels of the Earth from the beginning of the World to this Time.

Let QΠ expound any Sum of Money due to some Person at the End of a full Year. Now it is certain, that if the Debtor should pay down present the whole Sum of Money, he would lose the yearly Usury or Interest that his Money would gain him; and so a lesser Sum, being put out to Interest, will at the End of one Year together with the Interest thereof, be equal to the Sum of Money QΠ. Now this present Sum of Money, which together with the Interest thereof is equal to the Sum of Mony QΠ, is called the present Worth of the Money QΠ. Let AN be the Logarithm of the Ratio which the Principal has to the Sum of the Principal and Interest, that is, if the Principal be twenty times the yearly Interest, let AN be the Logarithm of the Number $1 + \frac{1}{20}$ or 1.05, and take QY equal to AN; then will AY be the Logarithm of the present Worth of the Money QΠ. For it is manifest, that the Money YZ put out to Interest, will at the End of one Year amount to the Money QΠ, and so to have the Logarithm of the present Worth thereof, or YZ, the Logarithm AN, must be taken from the Logarithm AQ, and there will remain the Logarithm AY of the present Worth, or YZ. But if the Sum ΠQ be not due till the End of two Years, then the Logarithm 2 AN must be substracted from the Logarithm AQ, and there will remain AV, the Logarithm of the present Worth, or of the Sum that must be paid down present for the Money QΠ due at the End of

two Years. For it is manifest, that the Money VX being put out to Interest, will at the End of two Years amount to the Sum of Money QΠ. By the same Reason, if the Sum QΠ, be not due until the End of three Years, the Logarithm 3 AN must be substracted from the Logarithm of QΠ, and the Remainder AS, shall be the Logarithm of the Number ST, or ST shall be the present Worth of the Sum QΠ due at the three Years end. And universally, if the Logarithm AN be multiplied by the Number of Years, at the End of which the Sum QΠ is due, and the Number produced be taken from the Logarithm AQ, then will the Logarithm of the present Worth of the Sum QΠ be had. And from hence it is manifest, if 5386500000 Pounds be due to some Society at the End of 600 Years, then would the present Worth of that vast Sum of Money be scarcely a Farthing

If the proportional Right Lines HG, EF, AB, CD, *Fig.* 4. are Ordinates to the Axis of the logarithmical Curve, and if their Ends FH, DB, be joined by Right Lines, which produced meet the Axis in the Points P and K, then the Right Lines GP, AK, will be always equal. For since GH:EF::AB:CD *it will be* as GH:Fs::AB:DR. But because of the equiangular Triangles PGH, HsF, as also KAB, BRD, we have PG:Hs::(GH:Fs::AB:DR::) KA:BR. And since the Consequents Hs, BR, are equal, the Antecedents PG, KA, shall be also equal. W. W. D.

If the Right Lines CD, EF, equally accede to AB, GH, so that the Point D at last may concide with B, and the Point F with H, then the Right Lines DBK, FHP, which did cut the Curve before, will be changed into the Tangents BT, HV. And the Right Lines AT, GV, will be always equal to each other, that is, the Portion of the Axis AT, or GV, intercepted between the Ordinate and the Tangent, which is called the Subtangent, will every where be a constant and given Length. And this is one of the chief Properties of the logarithmical Curve; for the different Species or Forms of those Curves are determined by the Subtangents.

The Logarithms or the Distances from Unity of the same Number, in two Logarithmical Curves of different Species, will be proportional to the Subtangents

gents of their Curves. For let HBD, SNY, *Fig.* 4, 5. be Curves, whose Subtangents are AT, MX, and let AB = MN = Unity; also DC = QY, then shall AC the Logarithm of the Number CD, in the Logarithmical Curve HD be to MQ, the Logarithm of the Number QY, (or of the said CD,) in the Curve SY, as the Subtangent AT is to the Subtangent MX. For let there be supposed an infinite Number of mean proportional Terms between AB, CD, or NM, QY, in the Ratio of AB to *ab*, or MN to *mn*; and since AB = MN, then will *ab* = *mn*, as also *bc* = *no*. And because the Number of proportional Terms in each Figure are equal, they do divide the Lines AC, MQ, into equal Numbers of Parts, the first of which A*a*, M*m*, and so the said Parts shall be proportional to their Wholes, that is, it will be as A*a* : M*m* : : AC : MQ. And because the Triangles TAB, B*cb*, are similar, (for the Part of the Curve B*b* nearly coincides with the Portion of the Tangent,) as also the Triangles XMN, N*on*, we have A*a*, or B*c* : *bc* : : TA : AB.

Also as *no*, or *bc* : N*o* : : MN, or AB : MX.

Where (by Equality of Proportion) it will be B*c* : N*o* : : TA : MX : : A*a* : M*m* : : AC : MQ; *which was to be demonstrated*. If AT be call'd *a*, since AB : AT : : *bc* : B*c*, then will $Bc = \frac{a \times bc}{AB}$.

Hence, if the Logarithm of a Number extreamly near Unity, or but a small matter exceeding it, be given, then will the Subtangent of the Logarithmical Curve be had. For the Excess *bc* is to the Logarithm B*c*, as Unity AB is to the Subtangent AT. Or even if there are any two Numbers nearly equal, their Difference shall be to the Difference of their Logarithms, as one of the Numbers is to the Subtangent. For Example; if the Increment *bc* be ,00000 00000 00001 02255 31945 60259, and B*c* or A*a* the Logarithm of the Number *a b* be ,00000 00000 00000 44408 92098 50062. Now if a fourth Proportional be found to the said two Numbers and Unity, *viz.* 43429 44819 03251, this Number will give the Length of the Subtangent AT, which is the Subtangent of the Curve expressing *Brigg's* Logarithms.

If a Sum of Money be put out to Interest on this Condition, That a propotional Part of the yearly Rate of Interest thereof be accounted every Moment of Time, *viz.* so, that at the End of the first Moment of Time, or indefinity small Particle of a Year, the Interest gotten thereby be proportional to that Time; which being added to the Principal, again begets Interest at the End of the second Moment of Time, and then the Principal, and this Interest become a Principal, and so on. It is requir'd to find the Amount of that Sum at the Years End. Let a be nearly the Interest of Unity, or of one Pound. Then if one whole Year, or 1 gives the Interest a, the indefinity small Particle of a Year Mm, will give the Interest M$m \times a$, proportional to Mm; and accordingly, if Unity be expounded by MN, the first Increment thereof shall be $no = \mathrm{M}m \times a$. This being granted, let a Logarithmical Curve be suppos'd to be described through the Points Nn, whose Axis is OMQ. Then in this Curve, if the Portion of the Axis MQ expresses the Time, the ordinate Q y will represent the Money proportionally increasing every Moment, to that Time. For if there be taken ml, &c. $=$Mm, the Ordinates lp, &c. shall be in a Series of of continual Proportionals in the Ratio of MN to mn, that is, they increase in the same Ratio as the Money doth.

Again, let the Right Line NX touch the Logarithmical Curve in N, and the Subtangent thereof MX shall be constant and invariable, and the small Triangle Non shall be similar to the Triangle XMN. But it has been prov'd, that the Increment $no = \mathrm{M}m \times a = \mathrm{N}o \times a$; and so $no : \mathrm{N}o :: \mathrm{N}o \times a : \mathrm{N}o :: a : 1$. But as no is to No, so shall NM be to MX. Wherefore it shall be be as a is to 1, so is NM, or 1, to $\mathrm{MX} = \frac{1}{a} =$ Subtangent.

Now if the nearly Rate of Interest be $\frac{1}{20}$ part of the Principal, or if $a = \frac{1}{20} = ,05$, then will $\mathrm{MX} = \frac{1}{a} = 20$.

Because in different Forms of Logarithms, the Logarithms of the same Number, are proportional to the Subtangents of their Curves: If MQ expresses the

the Time of a whole Year, or Unity, then ſhall QY be the Amount of the Money at the Year's End. And to find QY, ſay, as MX, or 20 is to 0.4342944, (which Number expounds the Subtangent of the Logarithmical Curve expreſſing *Brigg*'s Logarithms) ſo is one Year or Unity to a *Briggian* Logarithm, anſwering to the Number QY. This Logarithm will be found 0.0217147, and the Number anſwering to the ſame is 1.05127 = QY, whoſe Increment above Unity, or the Principal, exceeds the yearly Intereſt '05 but a ſmall matter. And ſo if the yearly Intereſt of 100 Pounds be 5 Pounds, the proportional yearly Intereſt, which is added to the Principal 100 at the end of each Particle of the Year, will amount only at the Year's End to 5 Pounds 2 Shillings and 6 $\frac{1}{2}$ Pence.

And if ſuch a Rate of Intereſt be requir'd, that every Moment a Part of it continually proportional to the encreaſing Principal be added to the Principal, ſo that at the Year's End an Increment be produc'd that ſhall be any given Part of the Principal, for Example, the $\frac{1}{20}$ Part, ſay, as the Logarithm of the Number 1.05 is to 1; that is, as 0.0211893 is to 1, ſo is the Subtangent 0.432944 to $\frac{1}{a} = 20.49$, and then will $a = \frac{1}{20,49} = .0488$. For if ſuch a Part of the Rate of Intereſt .0488 be ſuppoſed as anſwers to a Moment, that is, having the ſame Ratio to .0488 as a Moment has to a Year, and it be made as Unity is to that Part of the Rate of Intereſt, ſo is the Principal to the Momentaneous Increment thereof; then will the Money continually increaſing in that manner be augmented at the Years end the $\frac{1}{20}$ Part thereof.

CHAP.

CHAP. VI.

Of the Method by which Mr. Briggs *computed his Logarithms, and the Demonstration thereof.*

ALTHO' Mr. *Briggs* has no where describ'd the Logarithmical Curve, yet it is very certain that from the Use and Contemplation thereof, the Manner and Reason of his Calculations will appear. In any Logarithmical Curve HBD, let there be three Ordinates AB, *ab*, *qs*, nearly equal to one another; this is, let their Differences have a very small Ratio to the said Ordinates; and then the Differences of their Logarithms will be proportional to the Differences of the Ordinates. For since the Ordinates are nearly equal to one another, they will be very nigh to each other, and so the Part of the Curve B*s*, intercepted by them, will almost concide with a straight Line; for it is certain, that the Ordinates may be so near to each other, that the Difference between the Part of the Curve and the Right Line subtending it, may have to that Subtence, a Ratio less than any given Ratio. Therefore the Triangles B*cb*, B*rs*, may be taken for Right-lin'd, and will be equiangular. Wherefore, as *sr* : *bc* : : B*r* : B*c* : : A*q* : A*a*; that is, the Excesses of the Ordinates or Lines above the least, shall be proportional to the Differences of their Logarithms. And from hence appears the Reason of the Correction of Numbers and Logarithms by Differences and proportional Parts. But if AB

be

be Unity, the Logarithms of Numbers ſhall be proportional to the Differences of the Numbers.

If a mean Proportional be found between 1 and 10, or which is the ſame thing, if the Square Root of 10 be extracted, this Root or Number will be in the middle Place between Unity and the Number 10, and the Logarithm thereof ſhall be $\frac{1}{2}$ of the Logarithm of 10, and ſo will be given. If again, between the Number before found, and Unity, there be found a mean Proportional, which may be done in extracting the Square Root of the ſaid Number, this Number or Root will be twice nearer to Unity than the former, and its Logarithm will be one half of the Logarithm of that, or one fourth of the Logarithm of 10. And if in this Manner, the Square Root be continually extracted, and the Logarithms biſected, you will at laſt get a Number whoſe Diſtance from Unity ſhall be leſs than the $\frac{1}{1\ 00000\ 00000\ 00000}$ part of the Logarithm of 10. And after Mr. *Briggs* had made 54 Extractions of the Square Root, he found the Number 1.00000 00000 00000 12781 91493 20032 3442, and its Logarithm was 0.00000 00000 00000,05551 11512 31257 82702. Suppoſe this Logarithm to be to equal A*q* or B*r*, and let *qs* be the Number found by extracting the Square Root; then will the Exceſs of this Number above Unity, *viz.* *rs*=,00000 00000 00000. 12781 91493 20032 34.

Now by means of theſe Numbers the Logarithms of all other Numbers may be found in the following Manner: Between the given Number (whoſe Logarithm is to be found) and Unity, find ſo many mean Proportionals, (as above), till at laſt a Number be gotten ſo little exceeding Unity, that there be 15 Cyphers next after it, and a like Number of ſignificative Figures after thoſe. Let this Number be *ab*, and let the ſignificative Figures with the Cyphers prefixed before them denote the Difference *bc*. Then ſay, As the Difference *rs* is to the Difference *bc*, ſo is B*r* a given Logarithm, to B*c*, or A*a*, the Logarithm of the Number *ab*; which therefore is given. And if this Logarithm be continually doubled, the ſame Number

of

of Times as there were Extractions of the Square Root, you will at last have the Logarithm of the Number sought. Also by this Way may the Subtangent of the Logarithmical Curve be found, *viz.* in saying, As *rs* : B*r* : : AB, or Unity : AT, the Subtangent, which therefore will be found to be 0. 43429 44819 03251; by which may be found the Logarithms of other Numbers; to wit, if any Number NM be given afterwards as also its Logarithm, and the Logarithm of another Number sufficiently near to NM be sought, say, As NM is to the Subtangent XM, so is *no* the Distance of the Numbers to N*o* the Distance of the Logarithms. Now, if NM be Unity = AB, the Logarithms will be had by multiplying the small Differences *bc* by the constant Subtangent AT.

By this Way may be found the Logarithms of 2, 3, and 7, and by these the Logarithms of 4, 8, 16, 32, 64, &c. 9, 27, 81, 243, &c. as also 7, 49, 343, &c. And if from the Logarithm of 10 be taken the Logarithm of 2, there will remain the Logarithm of 5, so there will be given the Logarithms of 25, 125, 625, &c.

The Logarithms of Numbers compounded of the aforesaid Numbers, *viz.* 6, 12, 14, 15, 18, 20, 21, 24, 28, &c. are easily had by adding togerher the Logarithms of the component Numbers.

But since it was very tedious and laborious to find the Logarithms of the Prime Numbers, and not easy to compute Logarithms by Interpolation, by first, second and third, &c. Differences, therefore the great Men, Sir *Isaac Newton*, *Mercator*, *Gregory*, *Wallis*, and lastly, Dr. *Hally*, have published infinite converging Series, by which the Logarithms of Numbers to any Number of Places may be had more expediently and truer : Concerning which Series Dr. *Hally* has written a learned Tract, in the *Philosophical Transactions*, wherein he has demonstrated those Series after a new Way. and shews how to compute the Logarithms by them. But I think it may be more proper here to add a new Series, by means of which may be found easily and expeditiously the Logarithms of large Numbers.

Let z be an odd Number, whose Logarithm is sought; then shall the Numbers $z - 1$ and $z + 1$ be even

even, and accordingly their Logarithms, and the Difference of the Logarithms will be had, which let be called y: Therefore, also the Logarithm of a Number, which is a Geometrical Mean between $z-1$ and $z+1$ will be given, *viz.* equal to the half Sum of the Logarithms. Now the Series $y \times \frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5} + \frac{181}{15120z^7} + \frac{13}{25200z^9}$ &c. shall be equal to the Logarithm of the Ratio, which the Geometrical Mean between the Numbers $z-1$ and $z+1$, has to the Arithmetical Mean, *viz.* to the Number z.

If the Number exceeds 1000, the first Term of the Series $\frac{y}{4z}$ is sufficient for producing the Logarithm to 13 or 14 Places of Figures, and the second Term will give the Logarithm to 20 Places of Figures. But if z be greater than 10000, the first Term will exhibite the Logarithm to 18 Places of Figures; and so this Series is of great Use in filling up the Logarithms of the Chiliads omitted by *Briggs*. For Exple; It is required to find the Logarithm of 20001. The Logarithm of 20000 is the same as the Logarithm of 2 with the Index 4 prefix'd to it; and the Difference of the Logarithms of 20000 and 20002, is the same as the Difference of the Logarithms of the Numbers 10000 and 10001, *viz.* 0. 00004 34272 7687. And if this Difference be divided by $4z$, or 80004, the Quotient $\frac{y}{4z}$

shall be ---- ---- ---- ----	0. 00000 00005 42813
And if the Logarithm of the Geometrical Mean be added	4. 30105 17093 02416
to the Quotient, the Sum will	4. 30105 17098 45230

be the Logarithm of 20001. Wherefore it is manifest, that to have the Logarithm to 14 Places of Figure, there is no Necessity of continuing out the Quotient beyond six Places of Figures. But if you have a Mind to have the Logarithm to 10 Places of Figures only, as they are in *Ulaq*'s Tables, the two first Figures of the Quotient are enough. And if the Logarithms of the Numbers above 20000 are to be found

found by this Way, the Labour of doing then will mostly consist in setting down the Numbers. *Note*, This Series is easily deduced from that found out by Dr. *Hally*; and those who have a mind to be inform'd more in this Matter, let them consult his above-nam'd Treatise.

FINIS.

N. B. Most Treatises of this Kind not having their Faults (which are many) affix'd to them, and so the Reader's Progress very often retarded thereby; for avoiding this, by carefully reading the Book, and examining the Demonstrations over and over, we have found the following *Errata*, which the Reader is desir'd to correct.

ERRATA

IN the Preface. *Page* ii. *line* 14. *for* accused *read* accuse; *p.* iii. *l.* 9. *for* Reformations *r.* Reformation; *p.* vi. *l.* 17. *for* became *r.* become; *ibidem l.* 18. *for* Construction *r.* Constructions; *p.* vii. *l.* 9. *for* Demonstrations *r.* Demonstration; *ibid. l.* 29. *for* Number *r.* Lumber.

Page 5. *l.* 8. *for* which is *r.* and; *p.* 6. *l.* 3. *for* C *r.* G; *p.* 8. *l.* 32. *insert this Mark* ‡; *ibid. l.* 23. *dele* ‡; *p.* 12. *for* * 1. of this *in Margin*, *r.* * 8. of this; *p.* 14. *for* ‡ Def. 10. *in Margin*, *r.* ‡ 8. of this; *ibid. l.* 12. *for* r *r.* C; *p.* 18. *l.* 1. *for* BB *r.* CB; *p.* 19. *l.* 29. *for* Part *r.* Point; *p.* 21. *l.* 21. *this Mark* * *should be just after the Word* TAKE; *p.* 33. *l.* 14. *after the Period must be inserted this.* Again, because AC is parallel to BD, and BC falls upon them, the alternate Angles ACB, BCD are equal to one another; *p.* 34. *l.* 36. *for* DCB *r.* DCE; *p.* 39. *l.* 17. *for* ECF *r.* CEF; *ibid. l.* 25. *for* AB *r.* DB; *p.* 40. *l.* 35. *for* AMB *r.* ABM; *p.* 41. *against l.* 31. *in Margin r.* * 29. 1. *Ibid. for* ‡ 35, *in Margin r.* ‡ 34; *p.* 48. *for* † 41. 1. *r.* † 31. 1; *Ibid. against l.* 40, *read in the Margin* ‡ 29. 1; *p.* 49. *l.* 1. *for* GBC *r.* GCB; *p.* 50. *l.* 5. *for* KLG *r.* KLO; *Ibid. for* † 43. 1. *in Margin r.* † 36. 1; *page* 51. *against Line* 16. *for* † 36. 1. *in the Margin, read* † 31. 1; *Ibid. l.* 20. *for* ‡ *r.* †; *p.* 52. *against l.* 27. *in the Margin r.* † Cor. 4. of this; *p.* 55. *l.* 26. *for* EGB *r.* ECB; *p.* 57. *l.* 23. *for* ‡ *r.* *; *and against l.* 23 *in Margin r.* * 29. 1; *p.* 58. *in Margin for* ‡ 47 of this *r.* ‡ 47. 1; *p.* 60. *against l.* 35. *in Margin r.* * 12. 1; *p.* 66. *against l.* 23. *in Margin r.* ‡ Def. 20. 1; *p.* 68. *l.* 1. *for* CBG, *r.* ABC; *p.* 69. *l.* 31. *for* ‡ *r.* †; *p.* 79. *against l.* 12. *in Margin r.* † 4. 1; *p.* 73. *against l.* 34. *in Margin r.* * 10. 1; *p.* 75. *for* * 12. 1. *r.* * 20. 1; *p.* 77. *in the Margin, for* 4 of this *r.* 3 of this; *p.* 81. *in Margin against l.* 12. *for* † 32. 1. *r.* † 4. 1; *p.* 83. *l.* 15. *for* EFC *r.* FEC; *p.* 86. *against l.* 28. *in Margin r.* * 6. 1; *p.* 89. *l.* 18. *for* Base LF *r.* Base EF; *p.* 91. *l.* 21. *for* AEC *r.* ACE *; *Ibid. l.* 23. *dele* *; *p.* 93. *for* † 19. 1. *r.* † 19. 3; *p.* 94. *for* ‡ 4 of this, *in Margin r.* ‡ 4. 1; *Ibid. in Margin for* 31. *r.* 32; *Ibid. l.* 23. *for* AEL *r.* AEB; *p.* 97. *l.* 25. *dele this Mark* †; *Ib. l.* 28. *for this Mark* ‡ *r.* †; *Ib. l.* 34. *for* * *r.* †; *p.* 98. *l.* 16. *for* DA *r.* DB; *p.* 100. *l.* 16. *for* Sides, *r.* Angles; *p.* 103. *l.* 11. *for* * *r.* †; *p.* 104. *in the Margin for* 6. 3. *r.* 16. 3. *and* 30. 1. *r.* 10. 1; *p.* 105. *l. the last for* AB *r.* DB; *p.* 106. *for* * 31. 1. *r.* * 31. 3; *Ibid. l.* 6. *for* equiangular *r.* equilateral; *p.* 108. *l.* 19. *after the Word* Distances *r.* EA; *p.* 110. *l.* 36. *for* Triangle *r.* Pentagon; *p.* 111. *for* † 5. 1. *r.* † 8. 1; *p.* 115. *for* * 18. 1. *r.* 15. 1; *p.* 117. *l.* 11. *dele* the; *p.* 120. *l.* 12. *for* trilicate *r.* triplicate; *p.* 132. *for* * 6. *r.* * 7; *p.* 140. *l.* 7. *for* is *r.* is to; *p.* 142. *l.* 3. *instead of reading*, that Magnitude which has a greater Proportion to the same Magnitude, *r.* that Magnitude to which the same Magnitude has a greater Proportion; *p.* 149. *l.* 11. *for* EF *r.* EC; *p.* 151. *l.* 16. *for* ADE *r.* BDE; *p.* 152. *for* † 29. 3. *r.* † 29. 1; *p.* 153 *l.* 26. *for* CE *r.* CD; *p.* 162. *l.* 16 *for* BC *r.* BG; *p.* 166. *l.* 36. *for* CDE *r.* CDF; *p.* 168. *after* EF *l.* 12. *r.* and as BC to BG, so is the Triangle ABC to the Triangle ABG. Whence the Triangle ABC bears to the Triangle ABG a duplicate Proportion to what BC doth to EF; *Ibid. l.* 15. *for* EA. *r.* BA; *Ibid. in Margin, for* 21. 5. *r.* 22. 5; *p.* 172. *in Margin for* * 11. 8. *r.* * 11. 5; *p.* 175. *in Margin, for* ‡ Com. proceed, *r.* Lemma preceed; *Ibid. in Margin, for* ‡ 20. 5. *r.* ‡ 22. 5; *p.*

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187. *l.* 33. *for* B L, and GBL, *r.* G B, G B C; *p.* 193. *l.* 12. *for* C D, *r.* C B; *p.* 194. *l.* 17. *for* G H *r.* G E; *p.* 199. *l.* 16. *for* Right *r.* Right Angles; *p.* 200. *in Margin*, *for* * 31. 2. and ‡ 14. of this, *r.* * 3. 1. and † 4. of this; *p.* 202. *l.* 6. *dele* can; *p.* 206. *l.* 1. *for* Place *r.* Plane; *p.* 210. *l. the last but two. for* L E *r.* L X; *p.* 216. *l.* 22. *for* L T *r.* L Y; *p.* 221. *l.* 37. *for* Φ Y *r.* Ψ Y; *p.* 222. *l.* 1. *and* 12. *for* Φ Y *r.* Ψ Y; *p.* 223. *l.* 37. *for* E H *r.* F H; *p.* 224. *l.* 17. *for* E K *r.* to E K; *p.* 227. *in Margin*, *for* * 30. of this, *r.* * 31. of this; *p.* 228. *l.* 25. *for* B I *r.* B T; *p.* 234. *l.* 20. *for* plane Angle *r.* only plane; *p.* 235. *l.* 6. *after the Word* Diameter, *dele Comma, and r.* of; *p.* 236. *l.* 11. *for* which *r.* which is; *Ibid. in Margin*, *for* * 14. 1. *r.* * 41. 1; *p.* 238. *in Margin*, *for* † 22. 1. *r.* † 20. 6; *p.* 241. *l.* 31. *for* so is *r.* so is *; *Ib. in Margin*, *for* * 14. 5. *r.* * 11. 5; *p.* 242. *l.* 10. *for* into *r.* into two; *p.* 259. *l.* 5. *for* P E *r.* P F; *p.* 263. *l.* 1. *for* E M O N *r.* F M O N; *p.* 268. *l.* 23. *dele* not; *p.* 270. *l.* 16. *for* is to, *r.* is equal to; *Ibid. in Margin*, *for* † 23. 1. *r.* † 33. 1; *Ibid. l.* 38. *for* R B O S *r.* K B O S; *p.* 282. *for* the of *r.* the Sine of; *p.* 283. *l.* 5. *for* beging the *r.* beginning of the; *p.* 28 . *l.* 29. *for* (by 13-1) *r.* (by 22-3); *Ibid. l.* 31. *for* (by 22. 3) *r.* (13. 1); *p.* 285. *l.* 13. *for* GL = AB *r.* GL = AB; *Ibid. l.* 22. *for* Prop. 5. *r.* Prop. 4; *p.* 286. *l.* 10. *for* Prop. 5. *r.* Prop. 6; *p.* 287. *read the last Line thus*, Or, $2z \times \frac{z^2}{1.2.} - \frac{z^4}{1.2.3.4.} + \frac{z^6}{1.2.3.4.5.6.}$ &c. *p.* 288. *l.* 12. *for* $a \pm c$ *r.* $a = 0$; *Ibid. l.* 14. *for* 6. *r.* b; *p.* 291. *l.* 14. *dele* from; *p.* 294. *in the Title (and in the Titles afterwards) for* Spherical Triangles *r.* Spherical Trigonometry; *p.* 296. *l.* 26. *for* comprehending *r.* comprehended by; *p.* 301. *l.* 9. *for* nnarer *r.* nearer; *Ibid. l.* 22. *for* R S *r.* R S q; *p.* 302. *l.* 22. *for* the same *r.* a Different; *Ib. l.* 23. *for* a Different *r.* the same; *p.* 316. *l.* 32. *for* or *r.* and; *Ibid. l.* 35. *for* B L *r.* B D; *p.* 317. *l.* 16. *for* C D *r.* B D; *Ibid. in the Column under the Word* sought *against Number* 7. *for* the Side D C *r.* the Angle C; *Ibid. in the Column under the Word* given, *against Number* 8. *for* the Angle B C D *r.* the Angles B C D, and B; *Ibid. in the Column under the Word* sought, *against Number* 8, *for* Angles B C D *r.* the Side D C; *p.* 318. *l.* 4. *for* D B *r.* D C; *page* 319. *l.* 10. *for* P T R *r.* S T R; *p.* 320. *l.* 28. *for* another *r.* of another; *p.* 334. *l.* 18. *for* $1\frac{1}{2}$ *r.* $1\frac{1}{2}$; *p.* 335. *for* ΣΔLM *r.* ΣΠLM; *p.* 337. *l.* 15. *for* therefore *r.* thereof; *p.* 343. *l.* 16. *for* if *r.* if to; *p.* 346. *l.* 29. *for* [illegible] *r.* [illegible]; *Ib. l.* 34. *for* [illegible] *r.* [illegible]; *p.* 348. *l.* 31. *for* triplate *r.* triple; *p.* 350. *l.* 23. *for* and *r.* of.

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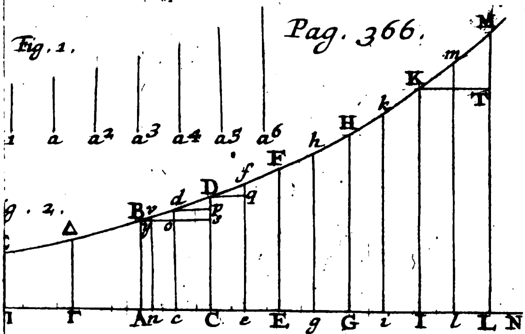

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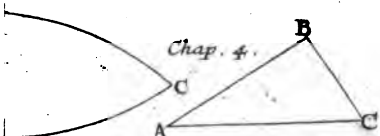
Fig. 2.



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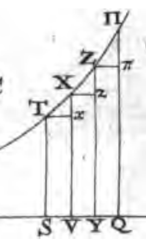

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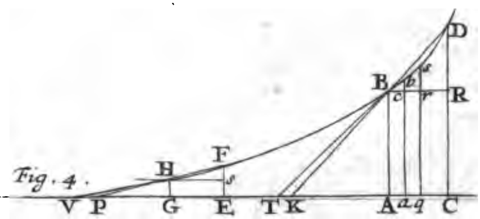

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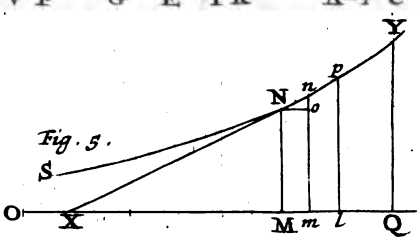

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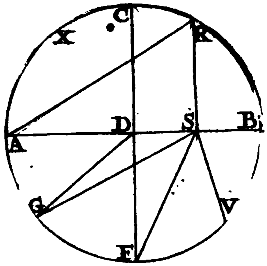

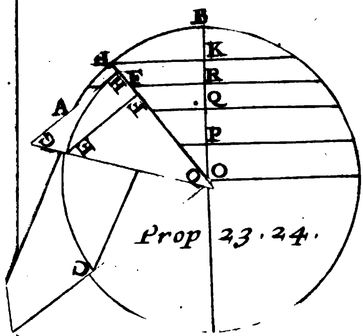

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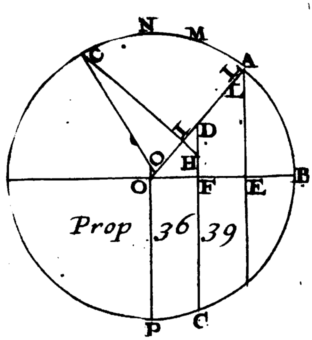

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